Computational approaches to the maximum number of distinct squares

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Outline

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Basic Notation

- A **square** is a repetition with power of 2, encoded as $(s, p, 2)$, **distinct squares** means only the types of the squares are counted, **primitively rooted distinct squares** means the generator itself is not a repetition. i.e. $x = aababaa$.
- $\boldsymbol{\sigma_d(n)}$ denotes the maximum number of primitively rooted distinct squares over all strings of length n containing exactly d distinct symbols.
- A **singleton** refers to a symbol in a string that occurs exactly once, a **pair** occurs exactly twice.

d-step Approach

We introduced a *d*-step approach to investigate the problem of distinct squares in relationship to the alphabet of the string [4].

d \ n - d	1	2	3	4	5	6	7	8	9	10	...
1	**1**	1	1	1	1	1	1	1	1	1	...
2	1	**2**	2	3	3	4	5	6	7	7	...
3	1	2	**3**	3	4	4	5	6	7	8	...
4	1	2	3	**4**	4	5	5	6	7	8	...
5	1	2	3	4	**5**	5	6	6	7	8	...
6	1	2	3	4	5	**6**	6	7	7	8	...
7	1	2	3	4	5	6	**7**	7	8	8	...
8	1	2	3	4	5	6	7	**8**	8	9	...
9	1	2	3	4	5	6	7	8	**9**	9	...
10	1	2	3	4	5	6	7	8	9	**10**	...
...	...	...	...	...	...	...	...	...	...	...	...

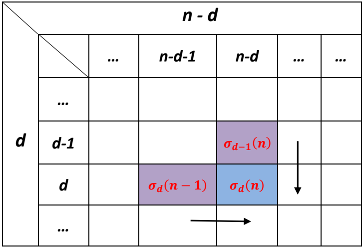
(d, n-d) Table: $\sigma_d(n)$ with $1 \le d \le 10$ and $1 \le n - d \le 10$

Outline

1. Introduction
2. *R*-cover
3. Computation on distinct squares
4. Program
5. Future work

Motivation

- Finding square-maximal strings by brute force algorithm is time consuming with large d and n.
- The goal is to prune down the search space of the square-maximal strings as much as possible.
- $(d,n\text{-}d)$ table shows $\sigma_d(n)$ is either larger than or equal to the previous computed values.



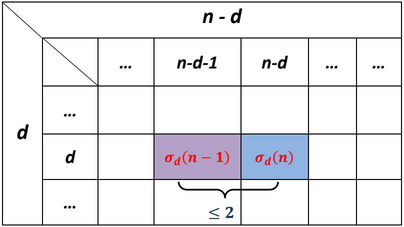
R-cover

We define a set of squares $\{R_i = (s_i, p_i, 2) : 1 \leq i \leq m\}$ is an R-cover of a string x if:

1. each R_i is a distinct primitively rooted square in x;
2. for every $R_i, 1 \leq i \leq m$, R_i is unique in $x[1..s_i + 2p_i - 1]$;
3. for every R_i and $R_{i+1}, 1 \leq i < m$, R_{i+1} is not in $x[s_i + 2p_i + 1..n]$;
4. for every R_i and $R_{i+1}, 1 \leq i < m$, $s_i < s_{i+1} \leq s_i + 2p_i$ and $s_i + 2p_i - 1 < s_{i+1} + 2p_{i+1} - 1$;
5. for all $1 \leq j \leq n$, there exists an $1 \leq i \leq m$ such that $s_i \leq j \leq s_i + 2p_i - 1$;

Double *R*-cover

- By Fraenkel-Simpson [1], there are at most two right most distinct squares starting at the same position in a string.
- $\sigma_d(n)$ is increased at most 2 compare to the previous computed value $\sigma_d(n-1)$.
- Further prune down the search space of square-maximal strings.



Double R-cover (cont.)

Proposition 1

A square-maximal string x with d symbols and length n has $\sigma_d(n) = \sigma_d(n-1) + 2$, x satisfy the **double R-cover density condition**: every letter in x occurs in at least two distinct squares.

Proof.

Suppose x does not meet the double R-cover density condition: there exist a letter in x occurs in only one distinct square. The removal of this letter (form string y), will destroy at most one square of x. Therefore $\sigma_d(n) = s(x) \leq s(y) + 1 \leq \sigma_d(n-1) + 1$ which leads to a contradiction. □

Outline

Along the row

- Motivation
  1. achieve better upper bound for $\sigma_d(n)$ with given d, i.e. $\sigma_2(n) \leq 2n - 47$ with $\sigma_2(37) = 27$;
  2. conjecture exact formula of $\sigma_d(n)$ with $d = 2$ row.
- Search space is narrowed down on strings satisfy double R-cover density.
- Compare to $\sigma_d(n-1)$, if rule out the possibility of $\sigma_d(n)$ increasing by 2 and by 1 in double R-cover strings search space, then search only one R-cover string that increases by 1 is sufficient.

Main diagonal

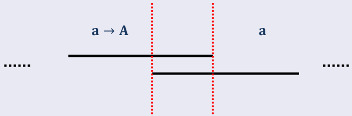
- Motivation
  1. prove the conjecture of $\sigma_d(n) \leq n - d$ is true up to certain d;
  2. achieve better upper bound of $\sigma_d(n)$ for all d and n, i.e. $\sigma_d(n) \leq 2n - d_0 - 2d$, where d_0 is the maximum where $\sigma_{d_0}(2d_0) = d_0$ is known;
- Computationally with smaller search space of strings:
  1. satisfy double R-cover density;
  2. contains at least $\lceil \frac{2d}{3} \rceil$ singletons[4], *i.e. computationally, we only need to generate strings with* $d = \lfloor \frac{d}{3} \rfloor$ *and* $n = d + \lfloor \frac{d}{3} \rfloor$;
  3. no pairs[4];
  4. parity condition.

Main diagonal (cont.)

Proposition 2

A square-maximal string x with d symbols and length $2d$ contains an R-cover satisfies the **parity condition**: the overlap of any two R-cover squares must contain the symbols that occur in both of the non-overlapping parts.

Proof.



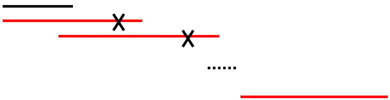
		n - d				
		...	*d-1*	*d*	...	...
d	...	...				
	d-1		$\sigma_{d-1}(2d-2)$			
	d		...	$\sigma_d(2d)$		
	...		$\sigma_{d+1}(2d)$		...	
	...					...

Suppose there exist such symbol a, change every occurrences of a to A in the left non-overlapping part to form y. Thus, $\sigma_d(2d) = s(x) \leq s(y) \leq \sigma_{d+1}(2d) = \sigma_{d-1}(2d-2)$, a contradiction. □

Outline

Description

- Build double R-cover strings:
  1. With all possible periods, consider all primitive generators and extend it to form the first square;
  2. Calculate the weak point by far;
  3. Build the next square within range which satisfies:
     - the generator is primitive;
     - is unique by far.
  4. Repeat step 2 and 3 and until reach desire length;
  5. When desire length is reached, ensure there is no weak point.



- Apply the appropriate conditions: i.e. parity condition when computing the main diagonal value.

Preliminary Result

- Program was implemented in C++, and run on Advol5 (8× Quad-Core AMD Opteron 8356) and Advol3 (16× Dual Core AMD Opteron 885) server.
- Along the row:
  - $\sigma_2(37)$ was computed in about 5 days;
  - did not return any result for weeks by brute force program.
- On the main diagonal:
  - $\sigma_{16}(32)$ (equivalent to $\sigma_5(21)$) was computed in about 1 hour and 40 minutes;
  - took weeks to run $\sigma_{10}(20)$ by brute force program.

Future work

Future work

- Use heuristic search on R-cover strings. i.e. apply conditions that likely to return square maximal strings with increase of 1.
- Current program recomputes all R-cover strings for length $n-1$ when computes length n. Develop a mechanism to be able to extend R-cover strings from shorter ones.
- Parallelize the program to speed up the computation.
- Because of the nature of the R-cover strings for distinct squares, we have to allow intermediate squares crossing the R-cover squares which is the bottleneck of the program.
- If the first and the last R-cover squares are the same for a given string and its reversal string, then we could avoid generating duplicated strings by restricting $p_1 \leq p_m$.

References

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- L. Ilie, *A simple proof that a word of length n has at most 2n distinct squares*, Journal of Combinatorial Theory Series A, 112, 1 (2005) 163-164.
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THANK YOU!