A d-step Approach for Distinct Squares in Strings

Mei Jiang
Joint work with Antoine Deza and Frantisek Franek

Advanced Optimization Laboratory
Department of Computing and Software
McMaster University

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Outline

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Background

- In 1998 Fraenkel and Simpson showed the number of distinct squares in a string of length n is bounded from above by $2n$ and gave a lower bound asymptomatically approaching n from below.
- In 2005 Ilie provided a simpler proof of Fraenkel and Simpson's main lemma and slightly improved the upper bound to $2n - \Theta(\log n)$ in 2007.
- It is believed, that the number of distinct squares is bounded by the length of the string.

d-step Approach

- We investigate the problem of distinct squares in relationship to the alphabet of the string.
- We construct a table whose rows are indexed by d and columns are indexed by $n - d$ with entries of $\sigma_d(n)$.
- We conjecture that the upper bound for the maximum number of primitively rooted distinct squares is $n - d$.
- d-step approach was inspired by the techniques used for investigating the Hirsch bound for the maximum possible diameter over all d-dimensional polytopes with n facets.

Basic Notation

- A **square** is a repetition with power of 2, **distinct squares** means only the types of the squares are counted, **primitively rooted distinct squares** means the generator itself is not a repetition.
- A **run**, a maximal fractional primitively rooted repetition, is formed by a maximal repetition followed by a tail.
- **s(x)** denotes the number of primitively rooted distinct squares in a string x.
- $\boldsymbol{\sigma_d(n)}$ denotes the maximum number of primitively rooted distinct squares over all strings of length n containing exactly d distinct symbols.
- A **singleton** refers to a symbol in a string that occurs exactly once, a **pair** occurs exactly twice, a **triple** occurs exactly three times, and in general an k-**tuple** (k times).

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$(d, n\text{-}d)$ Table Basic Properties

d \ n - d	1	2	3	4	5	6	7	8	9	10	...
1	**1**	*1*	1	1	1	1	1	1	1	1	...
2	1	**2**	*2*	3	3	4	5	6	7	7	...
3	1	2	**3**	*3*	4	4	5	6	7	8	...
4	1	2	3	**4**	*4*	5	5	6	7	8	...
5	1	2	3	4	**5**	*5*	6	6	7	8	...
6	1	2	3	4	5	**6**	*6*	7	7	8	...
7	1	2	3	4	5	6	**7**	*7*	8	8	...
8	1	2	3	4	5	6	7	**8**	*8*	9	...
9	1	2	3	4	5	6	7	8	**9**	*9*	...
10	1	2	3	4	5	6	7	8	9	**10**	...
...	...	...	...	...	...	...	...	...	...	...	...

(d, n-d) Table: $\sigma_d(n)$ with $1 \le d \le 10$ and $1 \le n-d \le 10$

For all $n \ge d \ge 2$:

1. $\sigma_d(n) \le \sigma_d(n+1)$
2. $\sigma_d(n) \le \sigma_{d+1}(n+1)$
3. $\sigma_d(n) < \sigma_{d+1}(n+2)$
4. $\sigma_d(n) = \sigma_{d+1}(n+1)$ for $n \le 2d$
5. $\sigma_d(n) \ge n-d$ for $n \le 2d$
6. $\sigma_d(2d) - \sigma_{d-1}(2d-1) \le 1$

Outline

Theorem 1

Theorem 1

For all $n \geq d \geq 2, \sigma_d(n) \leq n - d \iff \sigma_d(2d) = d$ for all $d \geq 2$

			n - d			
		...	...	d	...	...
d	...	...				
d	...		...			
d	d			d		
d	...			d	...	
d	...			d		...

Proof.

- $n < 2d$, constant under the diagonal.
- $n > 2d$, smaller or equal than the diagonal value.

$\square$

Theorem 2

Theorem 2

For all $n \geq d \geq 2, \sigma_d(n) \leq n - d \iff \sigma_d(2d{+}1) - \sigma_d(2d) \leq 1$ for all $d \geq 2$

		n - d				
		...	**d-1**	**d**	**...**	**...**
d	**...**	...				
	d-1		$\sigma_{d-1}(2d-2)$	$\sigma_{d-1}(2d-1)$	=	
	d		≤1	$\sigma_d(2d)$		
	...				...	
	...					...

Proof.

d is the least s.t. $\sigma_d(2d) > d$.
Remove the singleton,
$\sigma_{d-1}(2d-1) = \sigma_d(2d)$.
$\sigma_d(2d) - \sigma_{d-1}(2d-2) \leq 1$, and
$\sigma_{d-1}(2d{-}2) = d - 1$. Thus
$\sigma_d(2d) \leq d$. $\square$

Theorem 3

Theorem 3

For all $d \geq 2$, if $\sigma_d(2d{+}1) \leq d$, then

1. $\sigma_d(n) \leq n{-}d$ for all $n \geq d \geq 2$
2. $\sigma_d(n) \leq n{-}d{-}1$ for all $n > 2d \geq 4$

		n - d				
		…	**d**	**d+1**	…	…
d	…	…				
	d		$\mathbf{d}$	d		
	d+1		d	$\mathbf{d+1}$		
	…		d	$d+1$	…	
	…		d	$d+1$		…

Proof.

- $\sigma_d(2d) = \sigma_d(2d{+}1) = d$.
- $n > 2d$, smaller than the diagonal value.

$\square$

Theorem 4

Theorem 4

For all $d \geq 2$, if $\sigma_d(2d) = \sigma_d(2d{+}1)$, then

1. $\sigma_d(n) \leq n{-}d$ for all $n \geq d \geq 2$
2. $\sigma_d(n) \leq n{-}d{-}1$ for all $n > 2d \geq 4$

		n - d				
		...	d-1	d	...	...
d	...	...		=		
	d-1		$\sigma_{d-1}(2d-2)$	$\sigma_{d-1}(2d-1)$	≤1	
	d			$\sigma_d(2d)$		
	...				...	
	...					...

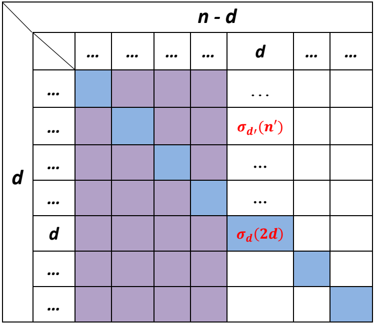
Proof.

To show $\sigma_d(2d) = \sigma_d(2d{+}1) = d$.
d is the least s.t. $\sigma_d(2d) > d$.
$\sigma_d(2d) - \sigma_{d-1}(2d-1) \leq 1$, and
$\sigma_{d-1}(2d{-}1) = d - 1$. Thus
$\sigma_d(2d) \leq d$. □

Outline

Relatively Short Square-Maximal Strings Structure

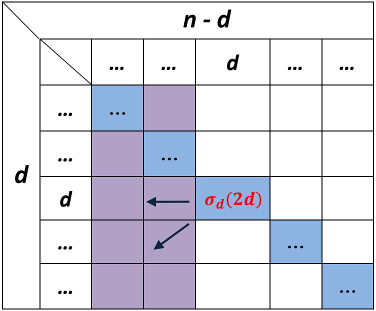
- We investigate the structure of square-maximal strings on the main diagonal.
  - If $\sigma_d(2d) = d$ then at least one of the square maximal string is in the form of $aabbccddeeff\ldots$
  - If $\sigma_d(2d) > d$ then the square maximal string is a counterexample. We investigate its structure and draw conclusions for counterexamples with $n \leq 4d$.



Pairs

Lemma 1

Let d is the least s.t. for some x, $s(x) = \sigma_d(2d) > d$. Then x does not contain a pair.



Proof.

The pair: $x[i_0] = x[i_1] = C$.

- Occurs in only one square. Replace the first C with a new symbol $\hat{C}$.
  $d - 1 \geq \sigma_{d+1}(2d) \geq \sigma_d(2d) - 1$.
- Occurs in a non-trivial run $uvCwuvCwu$. Remove wuv between C's.
  $d - k \geq \sigma_d(2d - k) \geq \sigma_d(2d) - k$, where $k = |w| + |u| + |v|$.

$\square$

Triples

Lemma 2

Let d is the least s.t. for some x, $s(x) = \sigma_d(2d) > d$. Then x can only contain a triple $x[i_0] = x[i_1] = x[i_2] = C$ that satisfies:

1. $x[i_0]$ and $x[i_1]$ occur in a run $r_1 = u_1 v_1 C w_1 u_1 v_1 C w_1 u_1$, where $|u_1| \geq 1$,
2. $x[i_1]$ and $x[i_2]$ occur in a run $r_2 = u_2 v_2 C w_2 u_2 v_2 C w_2 u_2$, where $|u_2| \geq 1$, and where $i_1 - i_0 \neq i_2 - i_1$,
3. either $u_1 v_1$ is a proper suffix of $u_2 v_2$, or $w_2 u_2$ is a proper prefix of $w_1 u_1$.

Triples (cont.)

Proof.

r_1 : $u_1 v_1 C w_1 u_1 v_1 C w_1 u_1$

r_2 : $u_2 v_2 C w_2 u_2 v_2 C w_2 u_2$

- Show it is impossible to have only two symbols occur in a run.
- Show it is impossible to have three symbols occur in the same run.
- Show it is impossible to have both ends are "long".

□

Singletons Estimation

Lemma 3

Let d is the least s.t. for some x, $s(x) = \sigma_d(2d) > d$. Then x has at least $\lceil \frac{2d}{3} \rceil$ singletons.

Proof.

- Let $u_1 v_1$ is a proper suffix of $u_2 v_2$, $a = u_1[0]$. a occurs at least 6 times in the r_1 and r_2. We assign 5 a's to the triple. It can be shown this assignment is mutually disjoint with others.

$$
\begin{array}{ll}
r_1 : & \dot{u}_1 \dot{v}_1 C w_1 \dot{u}_1 \dot{v}_1 C w_1 u_1 \\
r_2 : & \qquad\qquad u_2 v_2 C w_2 \underset{\bullet}{u_2} \underset{\bullet}{v_2} C w_2 u_2
\end{array}
$$

- m_0: the number of triples, m_1: the number of other multiply occurring symbols (at least 4 times), m_2: the number of singletons.

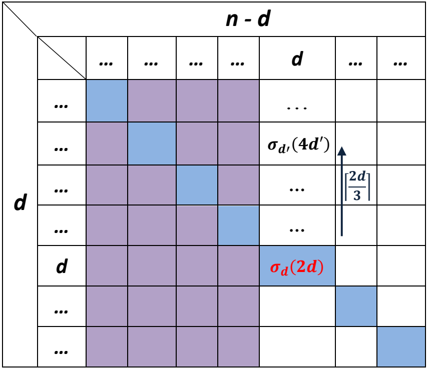
$$2d \geq 8m_0 + 4m_1 + m_2$$
$$d \leq 2m_0 + m_1 + m_2$$

Thus, $m_2 \geq \lceil \frac{2d}{3} \rceil$

Theorem 5

Theorem 5

For all $n \geq d \geq 2, \sigma_d(n) \leq n - d \iff \sigma_d(4d) \leq 3d$ for all $d \geq 2$



Proof.

d is the least s.t. $\sigma_d(2d) > d$.
Remove $\lceil \frac{2d}{3} \rceil$ singletons.
$\sigma_{d'}(4d') \geq \sigma_d(2d) > d$ and
$3d' = d$. Thus, $\sigma_{d'}(4d') > 3d'$. $\square$

Outline

Conclusions

- We exhibit the usefulness of investigating the main diagonal of $(d,n\text{-}d)$ table for tackling the conjectured upper bound.
  - To prove the conjecture by showing that the first counterexample has an impossible structure. i.e. it cannot contain an k-tuple, or if it contains an k-tuple, then it must contain another symbol with a frequency $> k$.
  - To disprove the conjecture by finding a counterexample on the diagonal.
- The Hirsch conjecture was recently disproved by Santos by exhibiting a violation on the diagonal with $d = 20$.
- Let's remark the techniques we used for "pushing up" the main diagonal can be applicable to the verification of the conjectured upper bound.

References

- A. S. Fraenkel and J. Simpson, *How Many Squares Can a String Contain?*, Journal of Combinatorial Theory Series A, 82, 1 (1998), 112-120.
- L. Ilie, *A simple proof that a word of length n has at most 2n distinct squares*, Journal of Combinatorial Theory Series A, 112, 1 (2005) 163-164.
- L. Ilie, *A note on the number of squares in a word*, Theoretical Computer Science, 380, 3 (2007), 373-376.
- F. Santos, *A counterexample to the Hirsch conjecture*, arXiv:1006.2814v1 (2010).
- A. Deza, F. Franek, and M. Jiang, *A d-step approach for distinct squares in strings*, AdvOL Technical Report 2011/01, Dept. of Computing and Software, McMaster University, Canada.

THANK YOU!