The Colourful Feasibility Problem

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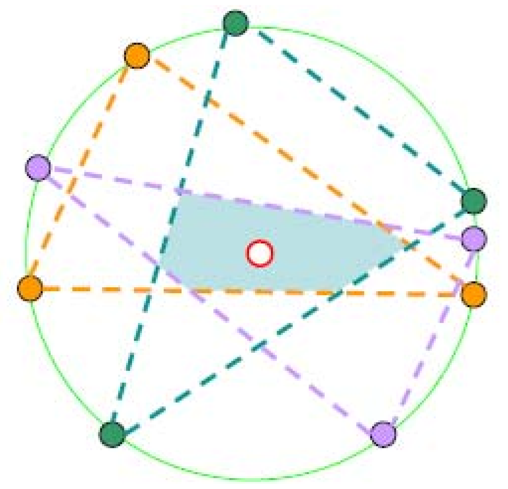
More info. at www.cas.mcmaster.ca/~deza/clp.html

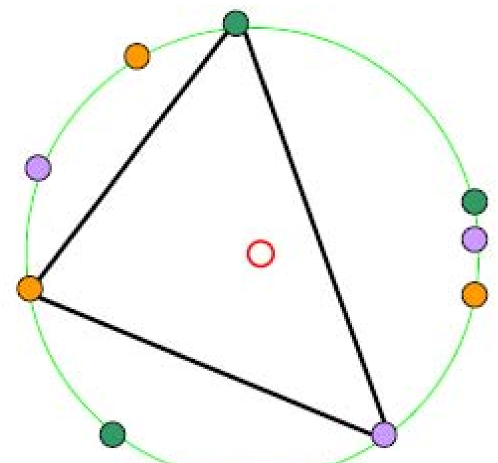
The problem

Given a set of points $S = \bigcup_{i=1}^{d+1} S_i$ and $o \in \bigcap_{i=1}^{d+1} \text{conv}(S_i)$ in $\mathbb{R}^d$, find a set $X = \{x_1, x_2, ..., x_{d+1}\}$, such that $o \in \text{conv}(X)$ and $\forall i (x_i \in S_i)$

Given Data:



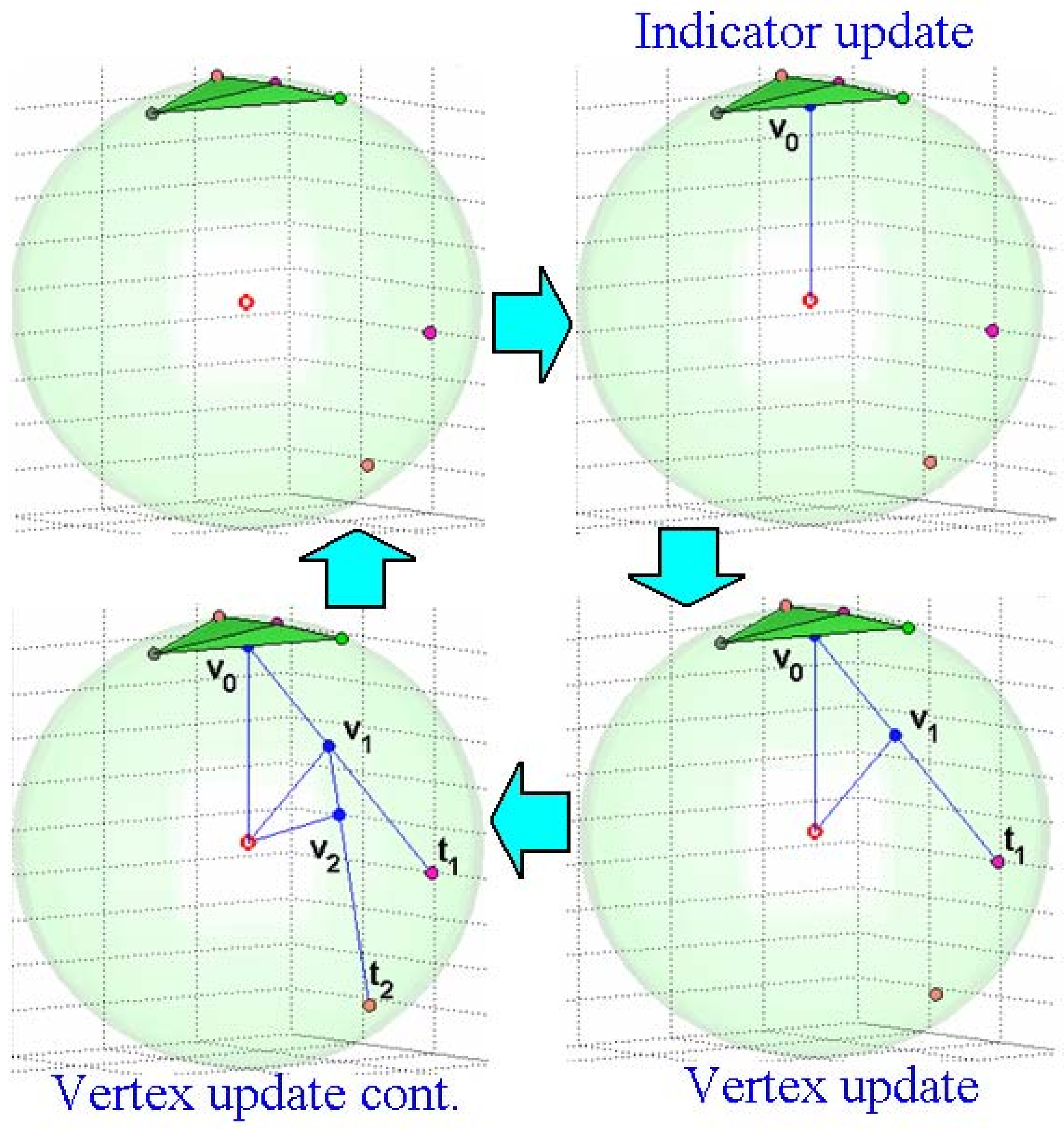
A feasible solution:



We assume that each coloured set S_i contains $d+1$ points and they are normalized.

A generalized algorithm

General idea: keep a simplex with vertices in different colours, and replace some vertices at each iteration; define a point v in the simplex, and reduce its norm at each iteration. v indicates how close the simplex is from a feasible solution.



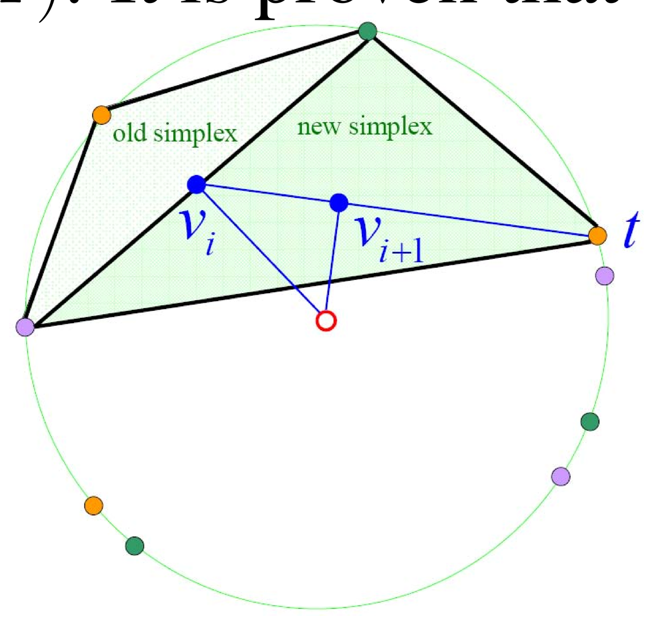
Initialization
  $X = \{x_1, ..., x_{d+1}\}$
  $v \in \text{boundary}(X)$
Vertex Update
  $I := \{\, i \mid v \in \text{conv}(X - \{x_i\}) \,\}$
  Forall $i \in I$
    $t := \arg\min_{x \in S_i} (v \bullet x)$;
    $x_i := t$; $v := \text{proj}_{vx_i}\, o$
Indicator Update
  If $o \notin \text{conv}(X)$
    Update v such that:
      - $\|v\|$ decreases
      - $v \in \text{boundary}(X)$;
    Goto Vertex Update
  Else terminate

Note: boundary(X) means the boundary of the convex hull of points in X

Convergence and complexity

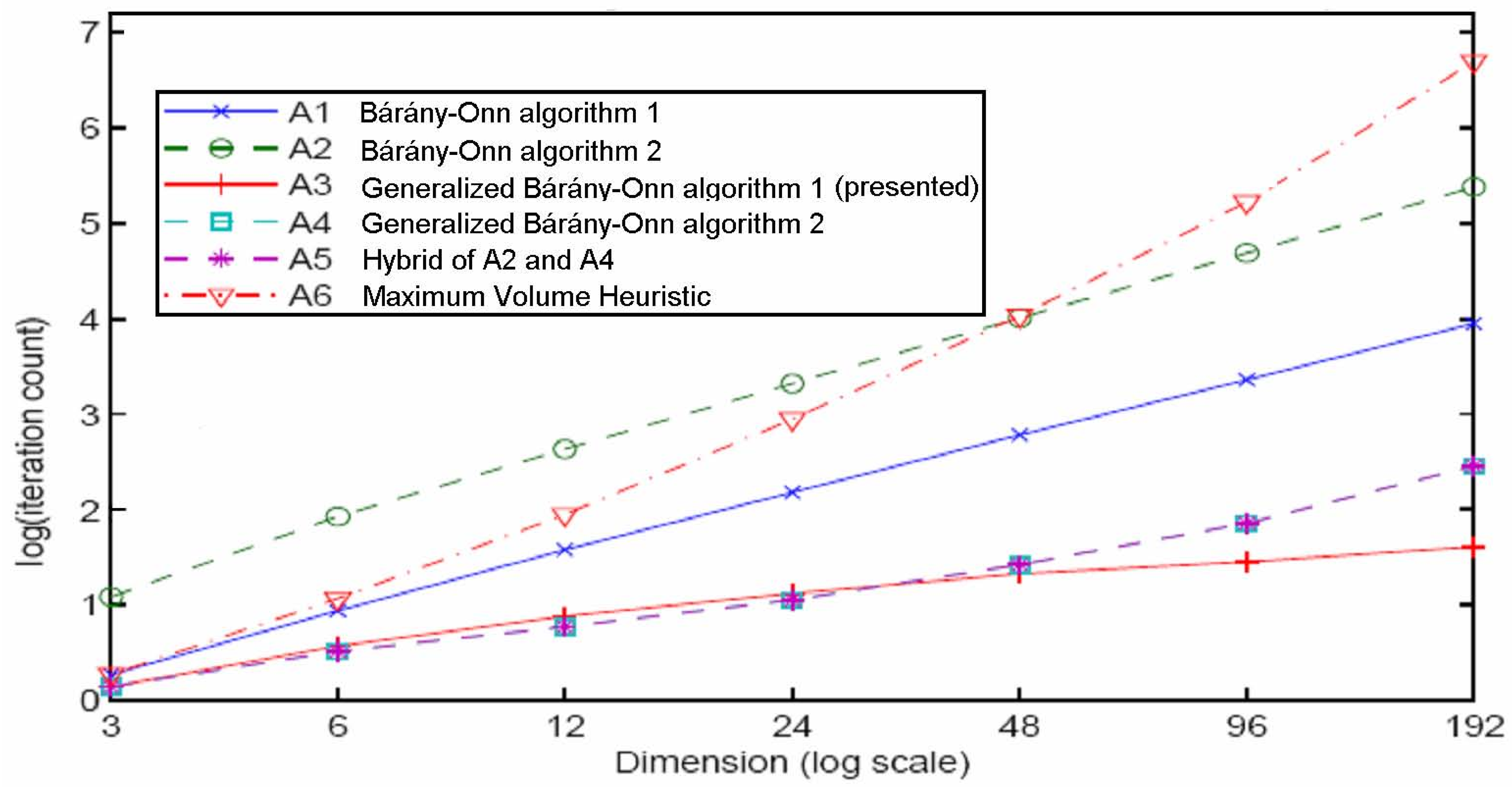
When $o \notin \bigcap_{i=1}^{d+1} \text{conv}(S_i)$, v_{i+1} is always on the interior of line segment between v_i and the vertex t coming into the colourful simplex. Hence, in the i-th iteration we have $\|v_{i+1}\| < \|v_i\|$ until $o \in \text{conv}(X)$. It is proven that the number of iterations to obtain an ϵ-close solution is $O\left(\frac{1}{\rho^2} \ln \frac{1}{\epsilon}\right)$ if $\rho > 0$, and $O\left(\frac{1}{\epsilon^2}\right)$ if $\rho = 0$.

ρ is the radius of a largest ball centered at the origin and inside $\bigcap_{i=1}^{d+1} \text{conv}(S_i)$



Test result

I implemented two algorithms proposed by Bárány and Onn, along with their generalizations and some heuristics such as the maximum volume heuristic. We used different types of randomly generated problems to test all these algorithms. The test result shows that the generalized Bárány-Onn algorithm 1 has the best performance.



Average number of iterations for one type of random problems.

Related research

Colourful simplicial depth (CSD) is the number of feasible solutions covering the given point. Cases that have CSD only d^2+1 were found, and it is proven that CSD is at least $\max(\lfloor \frac{(d+2)^2}{4} \rfloor, 3d)$. More results on CSD are expected.

Open questions

- Polynomially solvable?
- What if $o \notin \bigcap_{i=1}^{d+1} \text{conv}(S_i)$?
- Other efficient algorithms?

Main references

A. Deza, S. Huang, T. Stephen and T. Terlaky: *The colourful feasibility problem* (2006)

I. Bárány and S. Onn: *Colourful linear programming and its relatives*. Mathematics of Operational Research. (1997)