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On the generalized Berge sorting conjecture

Antoine Deza *, Feng Xie

McMaster University, Department of Computing and Software, Advanced Optimization Laboratory, Hamilton, Ontario, L8S 4K1, Canada

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ABSTRACT

In 1966, Claude Berge proposed the following sorting problem. Given a string of n alternating white and black pegs, rearrange the pegs into a string consisting of $\lceil\frac{n}{2}\rceil$ white pegs followed immediately by $\lfloor\frac{n}{2}\rfloor$ black pegs (or vice versa) using only moves which take 2 adjacent pegs to 2 vacant adjacent holes. Berge's original question was generalized by considering the same sorting problem using only *Berge k-moves*, i.e., moves which take k adjacent pegs to k vacant adjacent holes. The generalized Berge sorting conjecture states that for any k and large enough n, the alternating string can be sorted in $\lceil\frac{n}{2}\rceil$ Berge k-moves. The conjecture holds for $k=2$ and $n\geqslant 5$, and for $k=3$, $n\geqslant 5$, and $n\not\equiv 0 \pmod 4$. We further substantiate this conjecture by showing that it holds for $k=3$, $n\geqslant 20$, and $n\equiv 0 \pmod 4$. The introduced inductive solution generalized previous approaches and could provide insights to tackle the generalized Berge sorting conjecture.

1. Introduction

In a column that appeared in the Revue Française de Recherche Opérationnelle in 1966, entitled *Problèmes plaisans et délectables* in homage to the 17th century work of Bachet [2], Claude Berge [3] proposed the following sorting problem:

For $n\geqslant 5$, given a string of n alternating white and black pegs on a one-dimensional board consisting of an unlimited number of empty holes, we are required to rearrange the pegs into a string consisting of $\lceil\frac{n}{2}\rceil$ white pegs followed immediately by $\lfloor\frac{n}{2}\rfloor$ black pegs (or vice versa) using only moves which take 2 adjacent pegs to 2 vacant adjacent holes. Berge noted that the minimum number of moves required is 3 for $n=5$ and 6, and 4 for $n=7$. See Fig. 1 for a sorting of 5 pegs in 3 moves.

We believe that this problem was looked at within the last 40 years. For example, Shin-ichi Minato [7] found a solution in $\lceil\frac{n}{2}\rceil$ moves when he was a high-school student in 1981. However, the first published answer to Berge's question might have been given by Avis and Deza [1]. The Berge sorting question appeared in the *12th Prolog Programming Contest* [4] held in Seattle in 2006. In the statement of the problem, it is noted that this result is surprising given that initially half of the white pegs and half of the black pegs are incorrectly positioned. The following generalization displays an equally surprising pattern. Consider the same sorting problem using only *Berge k-moves*, i.e., moves which take k adjacent pegs to k vacant adjacent holes. After generating minimal solutions for a large number of k and n which turned out to be all equal to $\lceil\frac{n}{2}\rceil$ except for the few first small values of n, Deza and Hua [5] conjectured that, for n large enough, the minimum number of Berge k-moves to sort the alternating n-string is independent of k and equal to $\lceil\frac{n}{2}\rceil$. As the case $k=1$ is trivial and the case $k=2$ corresponds to the original Berge's question, the first case to investigate is $k=3$. A solution in $\lceil\frac{n}{2}\rceil$ Berge 3-moves was given in [5] for $n\geqslant 5$, and $n\not\equiv 0 \pmod 4$. We close the case $k=3$ by exhibiting a solution in $\lceil\frac{n}{2}\rceil$ moves for $n\geqslant 20$ and $n\equiv 0 \pmod 4$. The introduced inductive solution generalized previous approaches and could provide insights towards a solution for the generalized Berge sorting conjecture.

* Corresponding author.
E-mail addresses: deza@mcmaster.ca (A. Deza), xief@mcmaster.ca (F. Xie).

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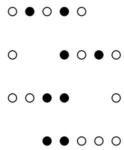

Fig. 1. Sorting 5 pegs in 3 moves.

2. Notation and previous results

2.1. Notation

We follow and adapt the notation used in [1,3,5]. The starting game board consists of n alternating white and black pegs sitting in the positions 1 through n. A single Berge k-move will be denoted as $\{j\ i\}$, in which case, the pegs in the positions $i, i+1, \ldots, i+k-1$ are moved to the vacant holes $j, j+1, \ldots, j+k-1$. Successive moves are concatenated as $\{j\ i\} \cup \{l\ k\}$, which means perform $\{j\ i\}$ followed by $\{l\ k\}$. Often, a move fills an empty hole created as an effect of the previous move, and the resulting notation $\{j\ k\} \cup \{k\ i\}$ is abbreviated as $\{j\ k\ i\}$. This can be extended to more than two such moves as well. Let $h(n,k)$ denote the minimum number of required k-moves, i.e., the length of a shortest solution, and $O_{n,k}$ denote an optimal solution for n pegs, i.e., a solution using $h(n,k)$ Berge k-moves. For example, we have $h(5,2) = 3$ and the optimal solution given in Fig. 1 is $O_{5,2} = \{6\ 2\ 5\ 1\}$. Up to symmetry, we can assume that the first move is to the right. Let $\mathcal{O}_{n,k}$ be the set of all optimal solutions starting with a move to the right. For example, there are 7 such optimal solutions in 10 Berge 3-moves to sort the alternating 20-string; that is, we have $h(20,3) = 10$ and $\mathcal{O}_{20,3} = \{\{21\ 2\ 7\ 12\ 17\} \cup \{24\ 13\ 22\ 6\ 1\} \cup \{17\ 8\ 24\},\ \{21\ 2\ 13\ 6\ 17\} \cup \{24\ 7\ 22\ 12\ 1\} \cup \{17\ 8\ 24\},\ \{21\ 2\ 13\ 8\ 17\} \cup \{24\ 6\ 22\ 12\ 1\} \cup \{17\ 7\ 24\},\ \{21\ 6\ 13\ 2\ 17\} \cup \{24\ 7\ 22\ 12\ 1\} \cup \{17\ 8\ 24\},\ \{21\ 8\ 13\ 2\ 17\} \cup \{24\ 6\ 22\ 12\ 1\} \cup \{17\ 7\ 24\},\ \{21\ 12\ 7\ 2\ 17\} \cup \{24\ 13\ 22\ 6\ 1\} \cup \{17\ 8\ 24\},\ \{21\ 16\ 3\ 10\} \cup \{24\ 17\ 22\ 5\ 1\} \cup \{10\ 15\ 6\ 24\}\}$. See Fig. 2 for an illustration of the optimal solution $O_{20,3} = \{21\ 2\ 7\ 12\ 17\} \cup \{24\ 13\ 22\ 6\ 1\} \cup \{17\ 8\ 24\}$.

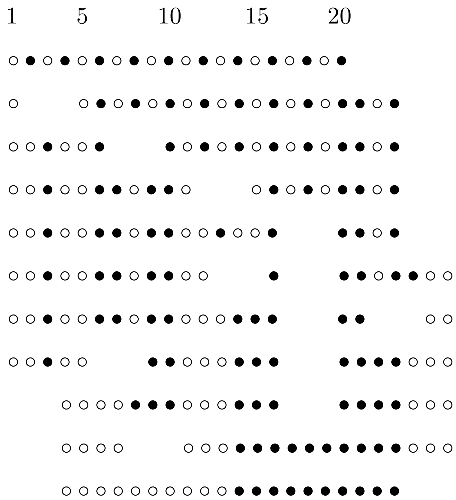

Fig. 2. Sorting the alternating 20-string in 10 Berge 3-moves.

2.2. Previous results

We recall the main results concerning the generalized Berge sorting conjecture.

Proposition 1. *(See [1,5].)*

(i) *For $n \geqslant 3$, we have $h(n,1) = \lceil \frac{n}{2} \rceil$ for $n \not\equiv 3 \pmod 4$ and $h(n,1) = \lfloor \frac{n}{2} \rfloor$ for $n \equiv 3 \pmod 4$.*
(ii) *For $n \geqslant 5$, we have $h(n,2) = \lceil \frac{n}{2} \rceil$.*
(iii) *For $n \geqslant 5$, we have $h(n,3) = \lceil \frac{n}{2} \rceil$ for $n \not\equiv 0 \pmod 4$.*
(iv) *For $4 \leqslant k \leqslant 14$ and $2k+11 \leqslant n \leqslant 50$, we have $h(n,k) = \lceil \frac{n}{2} \rceil$.*
(v) *For $n > k > 1$, we have $h(n,k) \geqslant \lceil \frac{n}{2} \rceil$.*

Conjecture 2. *(See [5].) For $k \geqslant 2$ and $n \geqslant 2k+11$, a string of n alternating white and black pegs can be sorted in $\lceil \frac{n}{2} \rceil$ Berge k-moves. In other words, $h(n,k) = \lceil \frac{n}{2} \rceil$ for $k \geqslant 2$ and $n \geqslant 2k+11$.*

3. Sorting the n-string in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \equiv 0 \pmod 4$

3.1. Inductive strategies

The solutions given in [5] are inductive and based on the following 3 step strategy.
IGNORE-AND-MAKEUP STRATEGY

Initialization. Find an optimal solution for the first values of n.
Induction from $n-p$ to n. Ignore a given number p of positions, use the moves corresponding to the solution $O_{n-p,3}$ to sort the alternating n-string.
Makeup. Reintroduce the ignored p positions and complete the sorting by $\lceil \frac{n}{2} \rceil - \lceil \frac{n-p}{2} \rceil$ additional moves.

For example, an optimal solution in $\lceil \frac{n}{2} \rceil$ Berge 3-moves can be obtained for $n \equiv 2 \pmod 4$ by the following version of the *ignore-and-makeup* strategy.

Initialization. We have $O_{6,3} = \{7\ 2\ 6\ 1\}$.
Induction from $n-4$ to $n = 4i+2 \geqslant 10$. Ignore the 4 positions $1, 2, 2i+3, 2i+4$ and use the moves corresponding to the solution $O_{n-4,3}$.
Makeup. Reintroduce the ignored positions and complete the sorting by the 2 additional moves $\{3\ 2i+4\ 1\}$.

The *ignore-and-makeup* strategy seems to fail for $n \equiv 0 \pmod 4$ and to be hard to extend to any k. We introduce the *ignore-and-knit* strategy which solves the case $n \equiv 0 \pmod 4$ and might be suitable to tackle the generalized Berge sorting conjecture. The *ignore-and-knit* strategy is also inductive but a bit more complicated as a move of the solution for $O_{n-p,3}$ is replaced by new *knitted* moves, and other makeup moves are added in the middle, instead of being added at the end as in the *ignore-and-makeup* strategy.

We introduced 3 features that characterize the optimal solution obtained by the *ignore-and-knit* strategy: *stage*, *pivot* and *anchor*. The moves of an optimal solutions obtained by the *ignore-and-knit* strategy are divided into 3 *stages*: stage (1) only alternating triples ○●○ or ●○● are moved, stage (2) only sorted bicolour triples ○○●, ●○○, ○●● or ●●○ are moved, and stage (3) only unicolour triples ●●● or ○○○ are moved. Another characteristic is that some pegs are never moved throughout the solution, and the corresponding positions are called *pivots*. The 3 positions emptied after the last move of stage (1) form the *anchor*. The anchor is surrounded by 2 pivots and remains empty throughout stage (2) before being filled by the first move of stage (3). To simplify the notation, let β be the position of the rightmost white peg after the last move; that is, without loss of generality, when $\lceil \frac{n}{2} \rceil$ white pegs are followed immediately by $\lfloor \frac{n}{2} \rfloor$ black pegs. As the optimal solutions obtained by the *ignore-and-knit* strategy for $n \equiv 0 \pmod 4$ are shifted to the right by 3 positions, we have $\beta = \frac{n}{2} + 3$. See Fig. 3 for an illustration of the 3 stages of the solution $O_{28,3}$, where $\beta = 17$, and the associated anchor corresponds to the positions $\beta-5$, $\beta-4$ and $\beta-3$ and the positions $\beta-12$, $\beta-6$, $\beta-2$, $\beta+3$ and $\beta+11$ are pivots.

3.2. Sorting the n-string in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \equiv 0 \pmod 4$

3.2.1. Sorting the n-string in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \equiv 4 \pmod 8$

The *ignore-and-knit* strategy for $n \equiv 4 \pmod 8$ and the solutions obtained are as follows:

Initialization. We have $O_{20,3} = \{21\ 2\ 7\ 12\ 17\} \cup \{24\ 13\ 22\ 6\ 1\} \cup \{17\ 8\ 24\}$, $O_{28,3} = \{29\ 2\ 7\ 16\ 23\ 12\} \cup \{32\ 17\ 30\ 25\ 21\ 6\ 1\} \cup \{12\ 23\ 8\ 32\}$, and $O_{36,3} = \{37\ 2\ 7\ 20\ 31\ 14\ 25\} \cup \{40\ 12\ 16\ 21\ 38\ 33\ 29\ 6\ 1\} \cup \{25\ 14\ 31\ 8\ 40\}$.
Induction and Knitting from $n-8$ to $n \geqslant 44$.
If $n \equiv 12 \pmod{16}$, ignore the 8 positions $\beta-6$, $\beta-5$, $\beta-4$, $\beta-3$, $\beta+4$, $\beta+5$, $\beta+9$, $\beta+10$. Use the moves of stage (1) of the solution $O_{n-8,3}$ followed by the additional move $\{\beta+6\ \beta-5\}$ to obtain stage (1) of the solution $O_{n,3}$. Use the moves of stage (2) of the solution $O_{n-8,3}$, except the penultimate move which is replaced by the 3 additional moves $\{\beta+12\ \beta+8\ \beta+4\ 6\}$ to obtain stage (2) of the solution $O_{n,3}$. Perform the additional move $\{\beta-5\ \beta+6\}$ before using the moves of stage (3) of the solution $O_{n-8,3}$ to obtain stage (3) of the solution $O_{n,3}$.
If $n \equiv 4 \pmod{16}$, ignore the 8 positions $\beta-9$, $\beta-8$, $\beta-4$, $\beta-3$, $\beta+4$, $\beta+5$, $\beta+6$, $\beta+7$. Use the moves of stage (1) of the solution $O_{n-8,3}$ followed by the additional move $\{\beta-7\ \beta+4\}$ to obtain stage (1) of the solution $O_{n,3}$. Use the moves of stage (2) of the solution $O_{n-8,3}$, except the first move which is replaced by the 3 additional moves $\{n+4\ \beta-9\ \beta-5\ \beta-17\}$ to obtain stage (2) of the solution $O_{n,3}$. Perform the additional move $\{\beta+4\ \beta-7\}$ before using the moves of stage (3) of the solution $O_{n-8,3}$ to obtain stage (3) of the solution $O_{n,3}$.

Some features of the solution $O_{n,3}$ for $n \equiv 4 \pmod 8$ and $n \geqslant 20$ are presented in Proposition 3. The additional move appended at the end of stage (1) creates the anchor which remains empty till the additional move inserted at the beginning

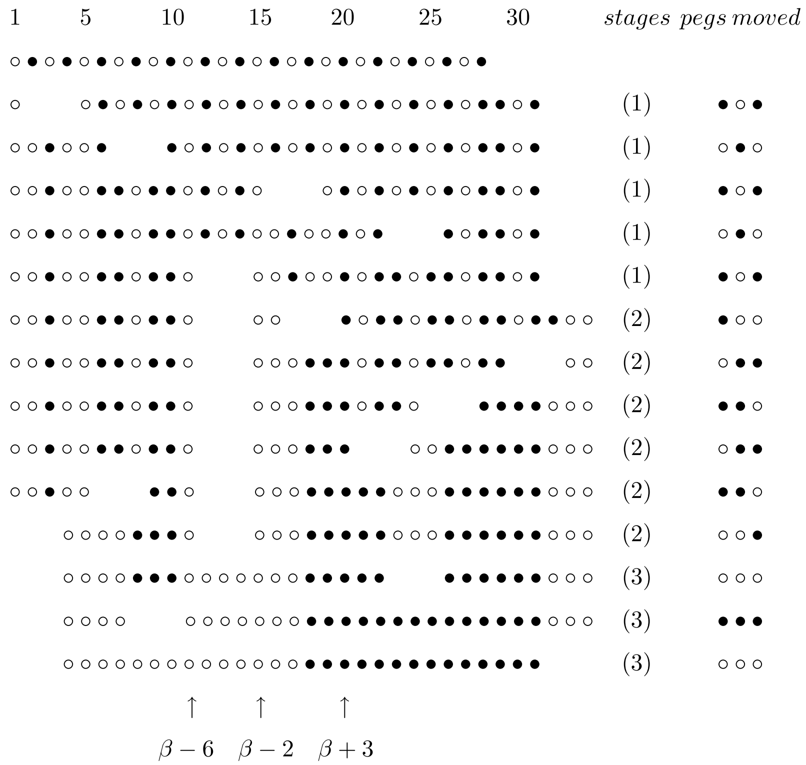

Fig. 3. Sorting the alternating 28-string by 14 Berge 3-moves.

of stage (3), and the 3 additional moves of stage (2) fit in the sequence of used moves of stage (1) and (2) of $O_{n-8,3}$. Note also that the ignored positions include or are followed or preceded by pivots. These observations and the features presented in Proposition 3 can be checked by induction. See Figs. 4 and 5 for an illustration of the features of the solutions $O_{n,3}$ for $n \equiv 4 \pmod 8$ where the jth move of stage i is numbered $(i.j)$. The 8 ignored positions for $n = 44$ are in red in Fig. 5. See Table 1 for an illustration of the induction from $O_{36,3}$ to $O_{44,3}$ for the *ignore-and-knit* strategy for $n \equiv 4 \pmod 8$. The additional moves (1.7), (2, 7), (2.8), (2.9) and (3.1) are bolded in Fig. 5 and Table 1. The optimal solutions $O_{n,3}$ for $n \equiv 4 \pmod 8$ and $n \leqslant 84$ are available online at [6].

Proposition 3. *The solutions $O_{n,3}$ for $n \equiv 4 \pmod 8$ and $n \geqslant 20$ obtained by the ignore-and-knit strategy satisfy the following properties*:

(i) *The solutions $O_{n,3}$ consist of $\lceil \frac{n}{2} \rceil$ Berge 3-moves and shift the string three positions to the right overall with the white pegs placed to the left of the black pegs.*

(ii) *If $n \equiv 4 \pmod{16}$, the positions $\beta - 2$, $\beta - 10$, $\beta + 3$ and $\beta + 7$ are pivots, and the positions $\beta + 4$, $\beta + 5$ and $\beta + 6$ form the anchor. If $n \equiv 12 \pmod{16}$, the positions $\beta - 6$, $\beta - 2$, $\beta + 3$ and $\beta + 11$ are pivots, and the positions $\beta - 5$, $\beta - 4$ and $\beta - 3$ form the anchor.*

(iii) *The first move of stage (2) places a triple $\bullet \circ \circ$ at the end of the string on the 3 positions starting from $\beta + \frac{n+2}{2}$. The last move $\{6\ 1\}$ of stage (2) places a triple $\circ \circ \bullet$ on the positions 6, 7 and 8.*

3.2.2. Sorting the n-string in $\lceil \frac{n}{2} \rceil$ Berge 3-moves for $n \equiv 0 \pmod 8$

The *ignore-and-knit* strategy for $n \equiv 0 \pmod 8$ and the solutions obtained are as follows:

Initialization. We have $O_{24,3} = \{25\ 6\ 13\ 18\} \cup \{-2\ 4\ 8\ 24\ 14\ 22\} \cup \{18\ 3\ 12\ -1\ 25\}$, $O_{32,3} = \{33\ 2\ 7\ 12\ 17\ 24\} \cup \{36\ 6\ 31\ 13\ 29\ 19\ 1\} \cup \{24\ 11\ 35\ 18\ 28\ 4\}$, $O_{40,3} = \{41\ 2\ 7\ 16\ 21\ 28\ 35\ 12\} \cup \{44\ 17\ 27\ 23\ 42\ 37\ 33\ 6\ 1\} \cup \{12\ 35\ 8\ 29\ 19\ 44\}$, and $O_{48,3} = \{49\ 2\ 7\ 20\ 25\ 32\ 43\ 14\ 37\} \cup \{52\ 12\ 16\ 22\ 31\ 27\ 49\ 45\ 41\ 6\ 1\} \cup \{37\ 14\ 43\ 8\ 33\ 23\ 52\}$.

Induction and knitting from $n - 8$ to $n \geqslant 56$.

If $n \equiv 8 \pmod{16}$, ignore the 8 positions $\beta - 12$, $\beta - 11$, $\beta - 10$, $\beta - 9$, $\beta + 10$, $\beta + 11$, $\beta + 15$ and $\beta + 16$. Use the moves of stage (1) of the solution $O_{n-8,3}$ followed by the additional move $\{\beta + 12\ \beta - 11\}$ to obtain stage (1) of the solution $O_{n,3}$. Use the moves of stage (2) of the solution $O_{n-8,3}$, except the penultimate move which is replaced by the 3 additional moves $\{\beta + 18\ \beta + 14\ \beta + 10\ 6\}$ to obtain stage (2) of the solution $O_{n,3}$. Perform the additional move $\{\beta - 11\ \beta + 12\}$ before using the moves of stage (3) of the solution $O_{n-8,3}$ to obtain stage (3) of the solution $O_{n,3}$.

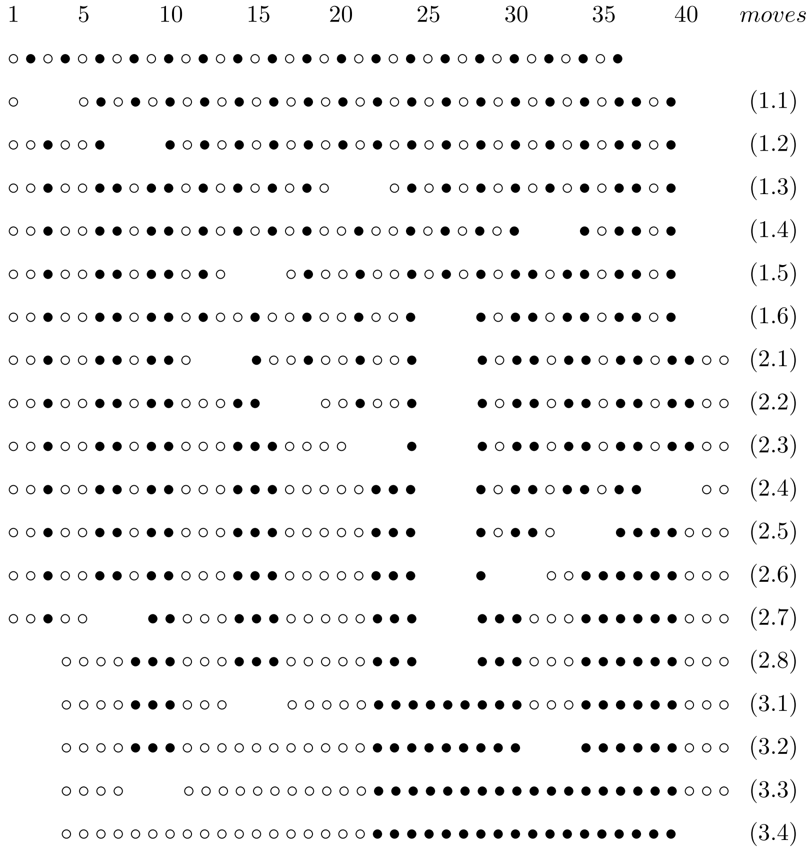

Fig. 4. Sorting the alternating 36-string by 18 Berge 3-moves.

If $n \equiv 0 \pmod{16}$, ignore the 8 positions $\beta-15$, $\beta-14$, $\beta-10$, $\beta-9$, $\beta+10$, $\beta+11$, $\beta+12$ and $\beta+13$. Use the moves of stage (1) of the solution $O_{n-8,3}$ followed by the additional move $\{\beta-13\ \beta+10\}$ to obtain stage (1) of the solution $O_{n,3}$. Use the moves of stage (2) of the solution $O_{n-8,3}$, except the first move which is replaced by the 3 additional moves $\{n+4\ \beta-15\ \beta-11\ \beta-23\}$ to obtain stage (2) of the solution $O_{n,3}$. Perform the additional move $\{\beta+10\ \beta-13\}$ before using the moves of stage (3) of the solution $O_{n-8,3}$ to obtain the stage (3) of the solution $O_{n,3}$.

The solutions $O_{n,3}$ for $n \equiv 4 \pmod 8$ and $n \equiv 0 \pmod 8$ have similar features, see [6] for the optimal solutions $O_{n,3}$ for $n \equiv 0 \pmod 8$ and $n \leqslant 88$. In particular we have:

Proposition 4. *The solutions $O_{n,3}$ for $n \equiv 0 \pmod 8$ and $n \geqslant 24$ obtained by the ignore-and-knit strategy satisfy the following properties*:

(i) *The solutions $O_{n,3}$ consist of $\lceil \frac{n}{2} \rceil$ Berge 3-moves and shift the string three positions to the right overall with the white pegs placed to the left of the black pegs.*
(ii) *If $n \equiv 8 \pmod{16}$, the positions $\beta-12$, $\beta-8$, $\beta+9$ and $\beta+17$ are pivots, and the positions $\beta-11$, $\beta-10$ and $\beta-9$ form the anchor. If $n \equiv 0 \pmod{16}$, the positions $\beta-16$, $\beta-8$, $\beta+9$ and $\beta+13$ are pivots, and the positions $\beta+10$, $\beta+11$ and $\beta+12$ form the anchor.*
(iii) *The first move of stage (2) places a triple $\bullet \circ \circ$ at the end of the string on the 3 positions starting from $\beta+\frac{n+2}{2}$. The last move $\{6\ 1\}$ of stage (2) places a triple $\circ \circ \bullet$ on the positions 6, 7 and 8.*

The solutions $O_{n,3}$ in $\lceil \frac{n}{2} \rceil$ Berge 3-moves obtained by the *ignore-and-knit* strategy for $n \equiv 0 \pmod 4$, and items (iii) and (v) of Proposition 1, yield that $h(n,3) = \lceil \frac{n}{2} \rceil$ for $n \geqslant 17$; combined with the values of $h(n,3)$ for $5 \leqslant n \leqslant 16$, see [6], we have:

Theorem 5. *For $n \geqslant 5$, $n \neq 12$ and $n \neq 16$, a string of n alternating white and black pegs can be sorted in $\lceil \frac{n}{2} \rceil$ Berge 3-moves. In other words, $h(n,3) = \lceil \frac{n}{2} \rceil$ for $n \geqslant 5$ except $h(12,3) = 7$ and $h(16,3) = 9$.*

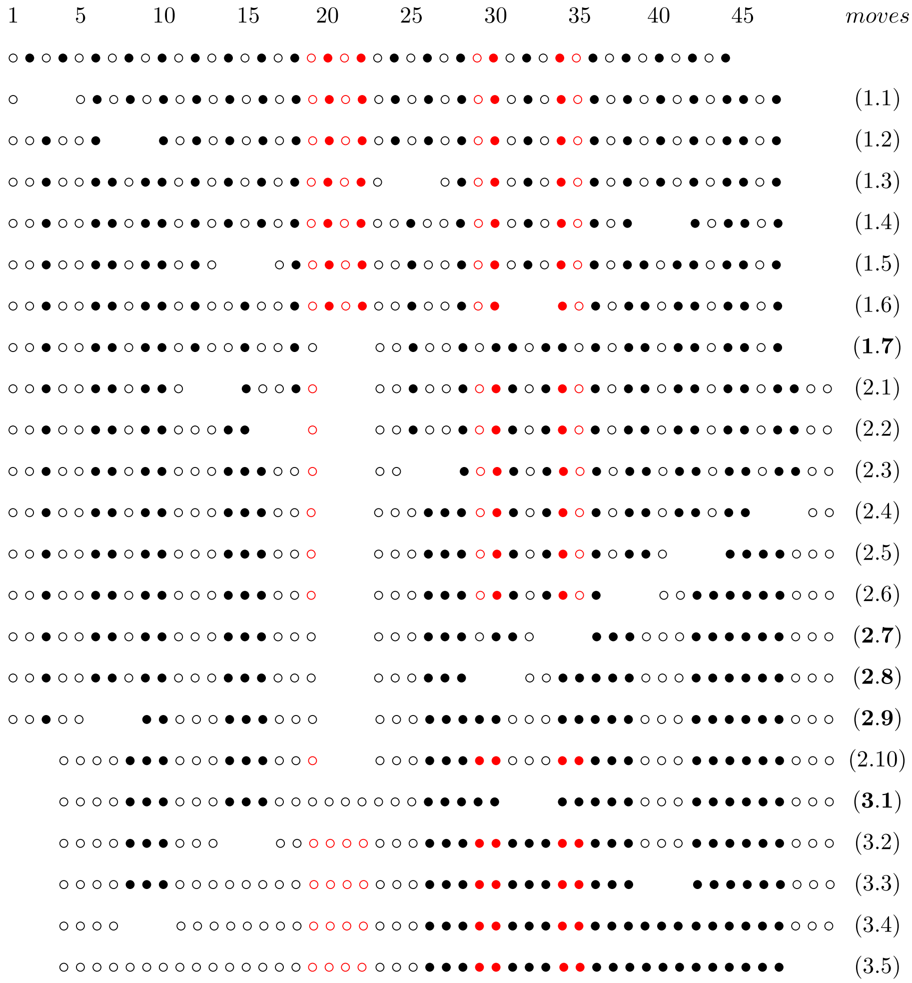

Fig. 5. Sorting the alternating 44-string by 22 Berge 3-moves. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Table 1
Induction from $O_{36,3}$ to $O_{44,3}$ for the *ignore-and-knit* strategy.

The 18 moves of $O_{36,3}$		The 22 moves of $O_{44,3}$	
(1.1)	$\{37\ 2\}$	$\{45\ 2\}$	(1.1)
(1.2)	$\{2\ 7\}$	$\{2\ 7\}$	(1.2)
(1.3)	$\{7\ \beta-1\}$	$\{7\ \beta-1\}$	(1.3)
(1.4)	$\{\beta-1\ \beta+10\}$	$\{\beta-1\ \beta+14\}$	(1.4)
(1.5)	$\{\beta+10\ \beta-7\}$	$\{\beta+14\ \beta-11\}$	(1.5)
(1.6)	$\{\beta-7\ \beta+4\}$	$\{\beta-11\ \beta+6\}$	(1.6)
–	–	$\{\beta+6\ \beta-5\}$	**(1.7)**
(2.1)	$\{40\ \beta-9\}$	$\{48\ \beta-13\}$	(2.1)
(2.2)	$\{\beta-9\ \beta-5\}$	$\{\beta-13\ \beta-9\}$	(2.2)
(2.3)	$\{\beta-5\ \beta\}$	$\{\beta-9\ \beta\}$	(2.3)
(2.4)	$\{\beta\ 38\}$	$\{\beta\ 46\}$	(2.4)
(2.5)	$\{38\ \beta+12\}$	$\{46\ \beta+16\}$	(2.5)
(2.6)	$\{\beta+12\ \beta+8\}$	$\{\beta+16\ \beta+12\}$	(2.6)
–	–	$\{\beta+12\ \beta+8\}$	**(2.7)**
(2.7)	$\{\beta+8\ 6\}$	$\{\beta+8\ \beta+4\}$	**(2.8)**
–	–	$\{\beta+4\ 6\}$	**(2.9)**
(2.8)	$\{6\ 1\}$	$\{6\ 1\}$	(2.10)
–	–	$\{\beta-5\ \beta+6\}$	**(3.1)**
(3.1)	$\{\beta+4\ \beta-7\}$	$\{\beta+6\ \beta-11\}$	(3.2)
(3.2)	$\{\beta-7\ \beta+10\}$	$\{\beta-11\ \beta+14\}$	(3.3)
(3.3)	$\{\beta+10\ 8\}$	$\{\beta+14\ 8\}$	(3.4)
(3.4)	$\{8\ 40\}$	$\{8\ 48\}$	(3.5)

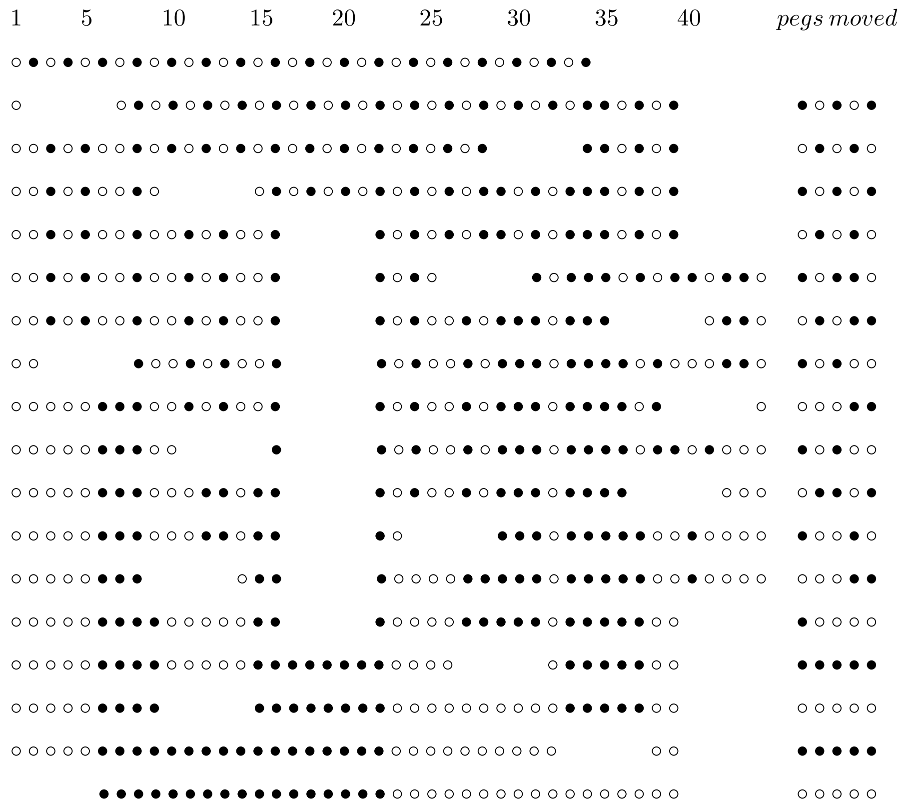

Fig. 6. Sorting the alternating 34-string by 17 Berge 5-moves.

4. Towards tackling the generalized Berge sorting conjecture

Many optimal solutions generated for larger k by computational search exhibit the features of the *ignore-and-knit* strategy. In particular, an anchor consisting of k consecutive positions which is emptied after stage (1) where only alternating k-tuples are moved, and remains empty till the first move of the last stage where only unicolour k-tuples are moved. The anchor is also surrounded by 2 pivots. See Fig. 6 for an illustration of the optimal solution $O_{34,5}$ sorting the alternating 34-string in 17 Berge 5-moves. While the complexity of knitting additional moves certainly increases with k, we believe that the *ignore-and-knit* strategy could yield insights toward solving the generalized Berge sorting conjecture.

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