Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Discrete Mathematics 31 (2008) 221–225
Electronic Notes in DISCRETE MATHEMATICS
www.elsevier.com/locate/endm

Diameter and Curvature: Intriguing Analogies

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Abstract

We highlight intriguing analogies between the diameter of a polytope and the largest possible total curvature of the associated central path. We prove continuous analogues of the results of Holt and Klee, and Klee and Walkup: We construct a family of polytopes which attain the conjectured order of the largest curvature, and prove that the special case where the number of inequalities is twice the dimension is equivalent to the general case. We show that the conjectured bound for the average diameter of a bounded cell of a simple hyperplane arrangement is asymptotically tight for fixed dimension. Links with the conjecture of Hirsch, Haimovich's probabilistic analysis of the shadow-vertex simplex algorithm, and the result of Dedieu, Malajovich and Shub on the average total curvature of a bounded cell are presented.

Keywords: Continuous d-step and Hirsch conjectures, polytopes, arrangements.

1 Polytopes: Diameter and Curvature

Let P be a full dimensional convex polyhedron defined by n inequalities in dimension d. The diameter $\delta(P)$ is the smallest number such that any two vertices of the polyhedron P can be connected by a path with at most $\delta(P)$ edges. The conjecture of Hirsch, formulated in 1957 and reported in [2], states that the diameter of a polyhedron defined by n inequalities in dimension d is

doi:10.1016/j.endm.2008.06.044

not greater than $n-d$. The conjecture does not hold for unbounded polyhedra. A polytope is a bounded polyhedron.

Conjecture 1.1 (Conjecture of Hirsch for polytopes) *The diameter of a polytope defined by n inequalities in dimension d is not greater than $n - d$.*

We consider a continuous analogue of the diameter introduced in [4]: the largest possible total curvature of the associated central path. We first recall the definitions of the central path and of the total curvature. For a polytope $P = \{x : Ax \geq b\}$ with $A \in \mathbb{R}^{n\times d}$, the central path corresponding to $\min\{c^T x : x \in P\}$ is a set of minimizers of $\min\{c^T x + \mu f(x) : x \in P\}$ for $\mu \in (0, \infty)$ where $f(x) = -\sum_{i=1}^{n} \ln(A_i x - b_i)$ is the standard logarithmic barrier function [14]. Intuitively, the total curvature [15] is a measure of how far off a certain curve is from being a straight line. Let $\psi : [\alpha, \beta] \to \mathbb{R}^d$ be a $C^2((\alpha - \varepsilon, \beta + \varepsilon))$ map for some $\varepsilon > 0$ with a non-zero derivative in $[\alpha, \beta]$. Denote its arc length by $l(t) = \int_{\alpha}^{t} \|\dot{\psi}(\tau)\| d\tau$, its parametrization by the arc length by $\psi_{\text{arc}} = \psi \circ l^{-1} : [0, l(\beta)] \to \mathbb{R}^d$, and its curvature at the point t by $\kappa(t) = \ddot{\psi}_{\text{arc}}(t)$. The total curvature is defined as $\int_0^{l(\beta)} \|\kappa(t)\| dt$. The requirement $\dot{\psi} \neq 0$ insures that any given segment of the curve is traversed only once and allows to define a curvature at any point on the curve. Let $\lambda^c(P)$ denote the total curvature of the central path corresponding to the linear optimization problem $\min\{c^T x : x \in P\}$. Considering the largest $\lambda^c(P)$ over all possible c we obtain the quantity $\lambda(P)$, referred to as the curvature of a polytope. The following continuous analogue of the conjecture of Hirsch was proposed in [4].

Conjecture 1.2 [4] *The order of the curvature of a polytope defined by n inequalities in dimension d is n.*

For polytopes and arrangements, respectively central path and linear optimization, we refer to Edelsbrunner [7], Grünbaum [11] and Ziegler [18], respectively Renegar [13], Roos et al [14] and Ye [16]. Holt and Klee [10] showed that, for $n > d \geq 13$, the conjecture of Hirsch is tight. Fritzsche and Holt [9] extended the result to $n > d \geq 8$. A family of d-dimensional polytopes $\mathcal{P}$ defined by $n > 2d$ non-redundant inequalities satisfying $\liminf_{n\to\infty} \lambda(\mathcal{P})/n \geq \pi$ for a fixed d was introduced in [4]. Thus, Conjecture 1.2 is the best possible.

Proposition 1.3 [4] *The conjectured order-n of the curvature is attained.*

The special case of $n = 2d$ of the conjecture of Hirsch is known as the d-step conjecture, and its continuous analogue is:

Conjecture 1.4 [5] *The order of the curvature of a polytope defined by $2d$ inequalities in dimension d is d.*

Klee and Walkup [12] showed that the d-step conjecture is equivalent to the conjecture of Hirsch. The continuous analogue of the result of Klee and Walkup. i.e., Conjecture 1.2 and Conjecture 1.4 are equivalent holds. Let $\Lambda(d,n)$ be the largest curvature $\lambda(P)$ over all polytopes P defined by n inequalities in dimension d.

Proposition 1.5 [5] *The continuous Hirsch conjecture is equivalent to the continuous d-step conjecture: If $\Lambda(d,2d) = \mathcal{O}(d)$ for all d, then $\Lambda(d,n) = \mathcal{O}(n)$.*

In Proposition 1.5 the constant 2 may be replaced by any integer $k > 1$, i.e., if $\Lambda(d,kd) = \mathcal{O}(d)$ for all d, then $\Lambda(d,n) = \mathcal{O}(n)$.

2 Arrangements: Diameter and Curvature

Let $\mathcal{A}$ be a simple arrangement formed by n hyperplanes in dimension d. We recall that an arrangement is called simple if $n \geq d+1$ and any d hyperplanes intersect at a unique distinct point. Since $\mathcal{A}$ is simple, the number of bounded cells (bounded connected component of the complement of the hyperplanes) of $\mathcal{A}$ is $I = \binom{n-1}{d}$. Following the approach of Dedieu, Malajovich and Shub [3], let $\lambda^c(\mathcal{A})$ denote the average value of $\lambda^c(P_i)$ over the bounded cells P_i of $\mathcal{A}$; that is, $\lambda^c(\mathcal{A}) = \sum_{i=1}^{i=I} \lambda^c(P_i)/I$. Note that each bounded cell P_i is defined by the same number n of inequalities, some being potentially redundant. Given an arrangement $\mathcal{A}$, the average curvature of a bounded cell $\lambda(\mathcal{A})$ is the largest value of $\lambda^c(\mathcal{A})$ over all possible c. Dedieu, Malajovich and Shub [3] demonstrated that $\lambda^c(\mathcal{A}) \leq 2\pi d$ for any fixed d

Proposition 2.1 [3] *The average curvature of a bounded cell of a simple arrangement defined by n inequalities in dimension d is not greater than $2\pi d$.*

Let $\delta(\mathcal{A}) = \sum_{i=1}^{i=I} \delta(P_i)/I$ denote the average diameter of a bounded cell of $\mathcal{A}$.

Conjecture 2.2 [4] *The average diameter of a bounded cell of a simple arrangement defined by n inequalities in dimension d is not greater than d.*

Let $\Delta_{\mathcal{A}}(d,n)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by n inequalities in dimension d. One can check that the bounded cells of a simple hyperplane arrangement $\mathcal{A}^*_{d,n}$ combinatorially equivalent to the cyclic hyperplane arrangement are mainly combinatorial cubes. Thus, the dimension d is an asymptotic lower bound for $\Delta_{\mathcal{A}}(d,n)$ for fixed d. Recall that the cyclic hyperplane arrangement is dual to the cyclic polytope, and see [8] for some combinatorial properties of the (projective) cyclic hyperplane arrangement.

Proposition 2.3 [6] *We have* $\Delta_{\mathcal{A}}(d,n) \geq d\binom{n-d}{d}/\binom{n-1}{d}$ *for* $n \geq 2d$.

3 Additional Links and Low Dimensions

Keeping the linear optimization approach over a simple arrangement but replacing central path following interior point methods by simplex methods, Haimovich's probabilistic analysis of the shadow-vertex simplex algorithm, see [1, Section 0.7], showed that the expected number of pivots is bounded by d. Note that while Dedieu, Malajovich and Shub consider only the bounded cells (the central path may not be defined over some unbounded ones), Haimovich considers the average over bounded and unbounded cells. While the result of Haimovich and Conjecture 2.2 are similar in nature, they differ in some aspects: Conjecture 2.2 considers the average over bounded cells, and the number of pivots could be smaller than the diameter for some cells.

Proposition 3.1 [4] *If the conjecture of Hirsch holds, then* $\Delta_{\mathcal{A}}(d,n) \leq \frac{d(n+1)}{n-1}$.

The arrangements resulting from the addition of one hyperplane to the cyclic hyperplane arrangement defined by $n-1$ inequalities are good candidates for achieving a large average diameter over the bounded cells. The combinatorics of the addition of a (pseudo) hyperplane to the cyclic hyperplane arrangement are studied in [17]. Considering such an arrangement with all the vertices on one side of the added hyperplane, one can show the following in dimension 2 and 3.

Proposition 3.2 [6] *For* $n \geq 4$, *we have* $\Delta_{\mathcal{A}}(2,n) = 2 - \lceil \frac{n}{2} \rceil / \binom{n-1}{2}$, *and* $3 - 6/(n-1) + (\lfloor \frac{n}{2} \rfloor - 2)/\binom{n-1}{3} \leq \Delta_{\mathcal{A}}(3,n) \leq 3 + 2(2n^2 - 16n + 21)/9\binom{n-1}{3}$.

Proposition 2.2 implies that $\lambda(P) \leq 2\pi d\binom{n-1}{d}$. For any polytope P, similarly to the proof of Proposition 1.5, c can be chosen large enough to force the central path to be almost orthogonal to a facet f of P, with the rest of the path resembling the central path corresponding to $\min\{c^T x : x \in f\}$. Thus, by induction, for any P and $\varepsilon > 0$, one can exhibit a c such that $\lambda^c(P) \geq (d-1)\pi/2 - \varepsilon$.

Proposition 3.3 *The curvature of a polytope defined by* n *inequalities in dimension* d *is between* $(d-1)\pi/2$ *and* $2\pi d\binom{n-1}{d}$.

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