Hyperplane Arrangements with Large Average Diameter

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By

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Abstract

This thesis deals with combinatorial properties of hyperplane arrangements. In particular, we address a conjecture of Deza, Terlaky and Zinchenko stating that the largest possible average diameter of a bounded cell of a simple hyperplane arrangement is not greater than the dimension. We prove that this conjecture is asymptotically tight in fixed dimension by constructing a family of hyperplane arrangements containing mostly cubical cells. The relationship with a result of Dedieu, Malajovich and Shub, the conjecture of Hirsch, and a result of Haimovich are presented.

We give the exact value of the largest possible average diameter for all simple arrangements in dimension two, for arrangements having at most the dimension plus two hyperplanes, and for arrangements having six hyperplanes in dimension three. In dimension three, we strengthen the lower and upper bounds for the largest possible average diameter of a bounded cell of a simple hyperplane arrangements.

Namely, let $\Delta_{\mathcal{A}}(d,n)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by n hyperplanes in dimension d. We show that

- $\Delta_{\mathcal{A}}(2,n) = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$ for $n \geq 3$,
- $\Delta_{\mathcal{A}}(d,d+2) = \frac{2d}{d+1}$,
- $\Delta_{\mathcal{A}}(3,6) = 2$,
- $3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)} \leq \Delta_{\mathcal{A}}(3,n) \leq 3 + \frac{4(2n^2-16n+21)}{3(n-1)(n-2)(n-3)}$,
- $\Delta_{\mathcal{A}}(d,n) \geq 1 + \frac{(d-1)\binom{n-d}{d} + (n-d)(n-d-1)}{\binom{n-1}{d}}$ for $n \geq 2d$.

We also address another conjecture of Deza, Terlaky and Zinchenko stating that the minimum number $\Phi^0_{\mathcal{A}}(d,n)$ of facets belonging to exactly one bounded cell of a simple arrangement defined by n hyperplanes in dimension d is at least $d\binom{n-2}{d-1}$. We show that

- $\Phi^0_{\mathcal{A}}(2,n) = 2(n-1)$ for $n \geq 4$,
- $\Phi^0_{\mathcal{A}}(3,n) \geq \frac{n(n-2)}{3} + 2$ for $n \geq 5$.

We present theoretical frameworks, including oriented matroids, and computational tools to check by complete enumeration the open conjectures for small instances. Preliminary computational results are given.

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Notations

A^T	transpose of matrix A
$I\!R^d$	d dimensional Euclidean space
$(I\!R^d)^*$	dual space of $I\!R^d$
$\mathbb{P}^d$	d dimensional projective space
$\mathbf{1}$	all 1 column vector
$\mathbf{0}$	all 0 column vector
$\mathbf{e_i}$	the unit vector along the i-th coordinate in $I\!R^d$
$\\|\mathbf{v}\\|$	the norm of vector $\mathbf{v}$
$\ddot{f}(t)$	the second derivative of function f with respect to variable t
$Dep(V)$	space of linear dependencies of a vector configuration
r-object	object of dimension r
$\mathcal{A}$	simple arrangement
$\mathcal{A}_{d,n}$	simple arrangement formed by n hyperplanes in $I\!R^d$
$\mathcal{A} \setminus \{h\}$	arrangement obtained by removing the hyperplane h from $\mathcal{A}$
$\mathcal{A} \cap h$	intersection of $\mathcal{A}$ with the hyperplane h
$f_k(P)$	number of k-faces of a polytope P
$f_k(\mathcal{A})$	number of k-faces of an arrangement $\mathcal{A}$
$f_k^+(\mathcal{A})$	number of k-faces that are inside the envelope of an arrangement $\mathcal{A}$
$f_k^0(\mathcal{A})$	number of k-faces that are on the envelope of an arrangement $\mathcal{A}$
$\delta(P)$	diameter of a polytope P
$\delta(\mathcal{A})$	average diameter of a bounded cell of $\mathcal{A}$
$\Delta_{\mathcal{A}}(d,n)$	largest possible average diameter of a bounded cell of $\mathcal{A}_{d,n}$
$\Phi^0_{\mathcal{A}}(d,n)$	smallest possible number of external facets of $\mathcal{A}_{d,n}$
$\mathcal{M}$	oriented matroid
$\mathcal{C}$	circuit of an oriented matroid
χ	chirotope
$\Lambda(n,r)$	ordered r-subsets of n vectors in $I\!R^r$

Chapter 1

Preliminaries

1.1 Polytopes

An *hyperplane* is the set $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}^T\mathbf{x} = c\}$ for some $\mathbf{a} \in \mathbb{R}^d$ $(\mathbf{a} \neq \mathbf{0})$ and $c \in \mathbb{R}$. If the equality is replaced with inequality, we have a *halfspace* , which is *closed* if the inequality is not strict. An hyperplane in $\mathbb{R}^d$ is isomorphic to $\mathbb{R}^{d-1}$.

Definition 1.1.1 *A polyhedron is an intersection of finitely many closed half-spaces. A polytope is a bounded polyhedron.*

Let P be a d-polyhedron, i.e., a polyhdron of dimension d. A closed halfspace is *valid* if P belongs to it. The hyperplane associated to a valid halfspace is called a *valid hyperplane.* A *face* of P is the intersection of P with some valid hyperplane. The 0-faces, 1-faces, $(d-2)$-faces and $(d-1)$-faces are called *vertices*, *edges*, *ridges* and *facets* respectively. The number of k-faces of P is denoted by $f_k(P)$ for $k = 0, \ldots, d-1$ and, considering the improper face P and the empty set, we have $f_d(P) = f_{-1}(P) = 1$.

Besides being represented as the intersection of closed halfspaces – *H-representation* – as in Definition 1.1.1, a polytope can also be represented as the convex hull of its vertices – *V-representation*. The conversion between H-representation and V-representation, also known as vertex enumeration or facet enumeration, is a well-studied problem in computational geometry.

1.2 Arrangements

Definition 1.2.1 *An arrangement $\mathcal{A}_{d,n}$ in $I\!R^d$ is a family of n ($n \geq d+1$) hyperplanes. The arrangement is simple if any d hyperplanes intersect at a distinct point.*

Definition 1.2.2 *A linear arrangement consists of hyperplanes containing the origin.*

Remark 1.2.3 *In a simple arrangement, no 2 hyperplanes are parallel to each other and no $d+1$ hyperplanes intersect at one point.*

In this thesis we consider mainly simple arrangements. The d-polyhedra defined by the hyperplanes of an arrangement $\mathcal{A}_{d,n}$ are called the d-faces or *cells* of $\mathcal{A}_{d,n}$. The k-faces of $\mathcal{A}_{d,n}$ are the k-faces of its cells. Let $f_k(\mathcal{A}_{d,n})$ denote the number of k-faces for $k = 0, \ldots, d-1$ and let $f_d(\mathcal{A}_{d,n})$ denote the number of cells of $\mathcal{A}_{d,n}$.

The bounded facets belonging to the unbounded cells are called *external*, and the facets belonging to 2 bounded cells are called *internal*. The k-faces belonging to an external facet, respectively internal facet, are called *external*, respectively *internal*, for $k = 0, \ldots, d-2$. Let $f_k^0(\mathcal{A}_{d,n})$ denote the number of external faces, and $f_k^+(\mathcal{A}_{d,n})$ the number of internal ones. The number of bounded cells is denoted by $f_d^+(\mathcal{A}_{d,n})$.

General references for polytopes and hyperplanes arrangements are the books of Edelsbrunner [14], Grunbaum [21] and Ziegler [38]. In the following, we recall some properties used in this thesis as well as few proofs.

Theorem 1.2.4 *For $k = 0, 1, \ldots, d$, a simple arrangement $\mathcal{A}_{d,n}$ has $f_k(\mathcal{A}_{d,n}) = \sum_{i=0}^{k} \binom{d-i}{k-i}\binom{n}{d-i}$ k-faces.*

Theorem 1.2.5 *A simple arrangement $\mathcal{A}_{d,n}$ has $f_d^+(\mathcal{A}_{d,n}) = \binom{n-1}{d}$ bounded cells.*

Proof. The proof is by induction on n and d. For $d = 1$ and $n \geq 2$, we have $f_d^+(\mathcal{A}_{d,n}) = n - 1 = \binom{n-1}{1}$. For $d \geq 1$ and $n = d + 1$, then $f_d^+(\mathcal{A}_{d,n}) = 1 = \binom{d+1-1}{d}$ as the only bounded cell is the d-simplex formed by the $d + 1$ hyperplanes. Assume that the statement holds for $\mathcal{A}_{d',n'}$ with $n' \leq n$ and $d' \leq d$ and one of the inequalities is strict. Then consider a simple arrangement $\mathcal{A}_{d,n-1}$ which, by hypothesis, has $\binom{n-2}{d}$ bounded cells. We add one hyperplane to $\mathcal{A}_{d,n-1}$ to get a simple arrangement $\mathcal{A}_{d,n}$. The intersection of the newly added hyperplane with $\mathcal{A}_{d,n-1}$ is a simple arrangement $\mathcal{A}_{d-1,n-1}$ which has, by hypothesis, $\binom{n-2}{d-1}$ $(d-1)$-dimensional bounded cells, that is, d-dimensional bounded facets, each of which gives rise to a new bounded cell on top of the existing $\binom{n-2}{d}$ bounded cells in $\mathcal{A}_{d,n-1}$. So we have $f_d^+(\mathcal{A}_{d,n}) = \binom{n-2}{d} + \binom{n-2}{d-1} = \binom{n-1}{d}$. □

Lemma 1.2.6 *A simple arrangement $\mathcal{A}_{d,n}$ has $n\binom{n-2}{d-1}$ bounded facets.*

Proof. For any hyperplane h_i of $\mathcal{A}_{d,n}$, $\mathcal{A}_{d,n} \cap h_i$ is a simple arrangement $\mathcal{A}_{d-1,n-1}$. By Theorem 1.2.5 we have $f^+_{d-1}(\mathcal{A}_{d,n} \cap h_i) = \binom{n-2}{d-1}$ and, as the bounded cells of $\mathcal{A}_{d,n} \cap h_i$ correspond to the bounded facets of $\mathcal{A}_{d,n}$, there are $n\binom{n-2}{d-1}$ bounded facets in $\mathcal{A}_{d,n}$.

Lemma 1.2.7 *A simple arrangement $\mathcal{A}_{d,n}$ has $\binom{n}{2}\binom{n-3}{d-2}$ bounded ridges .*

Proof. For each hyperplane h_i in $\mathcal{A}_{d,n}$ $(i = 1, 2, \ldots, n)$, $\mathcal{A}_{d,n} \cap h_i$ is a simple arrangement $\mathcal{A}_{d-1,n-1}$. By Theorem 1.2.6, each $\mathcal{A}_{d,n} \cap h_i$ has $(n-1)\binom{n-3}{d-2}$ $(d-1)$-dimensional bounded facets, or d-dimensional bounded ridges in $\mathcal{A}_{d,n}$. As each bounded ridge belongs to exactly 2 hyperplanes, we have $\frac{1}{2}n(n-1)\binom{n-3}{d-2}$ bounded ridges. □

Roughly speaking, a *projective arrangement* is an arrangement in the projective space $\mathbb{P}^d$ with one of the hyperplanes being at infinity. A projective arrangement $\mathcal{A}_{d,n}$ is *near trivial* if there is a point of $\mathbb{P}^d$ that is contained in all hyperplanes of $\mathcal{A}_{d,n}$ but one. We recall a fundamental result of Shannon [34].

Theorem 1.2.8 *Let $\mathcal{A}_{d,n}$ be a projective arrangement which is not near trivial and let h be a hyperplane of $\mathcal{A}_{d,n}$. Then there are at least $n-d-1$ simplicial d-cells having no facet in h.*

Corollary 1.2.9 *A simple arrangement $\mathcal{A}_{d,n}$ has at least $n-d$ simplex cells.*

Proof. By adding an hyperplane h_{n+1} at infinity to a simple (Euclidean) arrangement $\mathcal{A}_{d,n}$, we obtain a not near trivial projective arrangement $\mathcal{A}_{d,n+1}$. Applying Shannon's Theorem 1.2.8 with $h = h_{n+1}$, we get $n+1-d-1 = n-d$ simplicial cells in $\mathcal{A}_{d,n+1}$ having no facet in h_{n+1}, i.e., $n-d$ (bounded) simplex cells in $\mathcal{A}_{d,n}$. □

1.3 Fans

Definition 1.3.1 *A fan in $I\!R^d$ is a set of nonempty polyhedral cones satisfying:*

(1) Every nonempty face of a cone in the fan is also a cone in the fan.

(2) The intersection of any two cones in the fan is a face of both.

A *complete fan* is a fan that covers the whole space. A linear hyperplane arrangement decomposes the space into a complete fan. A *normal fan* of a polytope is a complete fan of which each cone is the set of linear functionals which are maximal on some face of the polytope. Followed is the formal definition.

Definition 1.3.2 *Let P be nonempty polytope in $I\!R^d$. The normal fan of P is the set of cones $\{N_F \mid F\, is\, a\, nonempty\, face\, of\, P\}$, where*

$$N_F := \left\{\mathbf{c} \in (I\!R^d)^* \mid F \subseteq \{\mathbf{x} \in P \mid \mathbf{c} \cdot \mathbf{x} = max\, \mathbf{c} \cdot \mathbf{y} : \mathbf{y} \in P\}\right\}.$$

1.4 Zonotopes and linear arrangements

In Euclidean space, The *Minkowski sum* of two sets P and Q is the set resulting from adding any point in P to any point in Q. i.e.,

$$P + Q := \{\mathbf{p} + \mathbf{q} \mid \mathbf{p} \in P, \mathbf{q} \in Q\}. \tag{1.4.1}$$

Definition 1.4.1 *A zonotope is the image of a cube under an affine projection $\pi : I\!R^s \to I\!R^d$, where $\pi(\mathbf{x}) = P\mathbf{x} + \mathbf{z}$, $\left(\mathbf{x} \in I\!R^s, \mathbf{z} \in I\!R^d, P \in I\!R^{d \times s}\right)$.*

Let $C_s = \{\mathbf{x} \in I\!R^s \mid -1 \leq x_i \leq 1, i = 1, 2, \cdots, s\}$ denote an s-cube and write $P = [\mathbf{p_1}, \mathbf{p_2}, \cdots, \mathbf{p_s}]$. Then a zonotope can be represented as the set

$$\begin{aligned} \pi(C_s) &= \{P\mathbf{x} + \mathbf{z} \mid \mathbf{x} \in C_s\} \\ &= \left\{ \mathbf{y} \in I\!R^d \mid \mathbf{y} = \sum_{i=1}^{s} x_i \mathbf{p_i} + \mathbf{z}, -1 \leq x_i \leq 1 \right\}. \end{aligned}$$

One can see from the above equation that a zonotope can also be represented as a shifted Minkowski sum of finitely many line segments.

Zonotopes have many interesting properties. For example, a zonotope is centrally symmetric and every face of a zonotope is again a zonotope. Please refer to Appendix A.2 for a description of the *permutahedron* of order 4 which is a 3 dimensional zonotope.

There is a correspondence between zonotopes and linear arrangements, which is exploited in the cell enumeration algorithm described in Section 6.2. The correspondence is as follow:

Proposition 1.4.2 *Given a set of line segments and the zonotope that is formed by the Minkowski sum of the line segments, the normal fan of the zonotope is the fan of the linear arrangement of the hyperplanes whose normal vectors are associated with the line segments.*

The proof of Proposition 1.4.2 is omitted, see Chapter 7 of [38].

1.5 Matroids and oriented matroids

Oriented matroids are a powerful abstract structure that can describe the combinatorial structure of hyperplane arrangements, as well as other discrete objects, including directed graphs, vector configuration, point configuration.

1.5.1 Matroids

We first give a brief introduction to matroids. A thorough introduction into this field can be found in [29]. The theory of matroids arises from the study of dependency in linear algebra and graphs, and many terms are borrowed from there.

Definition 1.5.1 (first definition) *A matroid $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a collection of subset of E satisfying the following three conditions:*

(I1) $\phi \in \mathcal{I}$.

(I2) *If $X \in \mathcal{I}$ and $Y \subset X$, then $Y \in \mathcal{I}$.*

(I3) *If $X, Y \in \mathcal{I}$ and $|Y| < |X|$, then there exists an element $e \in X \setminus Y$ such that $Y \cup \{e\} \in \mathcal{I}$.*

The collection $\mathcal{I}$ is called *independent sets* and the three conditions are referred to as *independent sets axioms*. The Axiom *(I3)* is called *independence augmentation axiom*.

A matroid can also be defined in terms of minimal dependent sets, or *circuits* .

Definition 1.5.2 (second definition) *A matroid $\mathcal{M}$ is an ordered pair $(E, \mathcal{C})$ where E is a finite set and $\mathcal{C}$ is a collection of subsets of E satisfying the following three conditions:*

(C1) $\phi \notin \mathcal{C}$.

(C2) If $X \in \mathcal{C}$ and $Y \subset X$, then $Y \notin \mathcal{C}$.

(C3) If $X, Y \in \mathcal{C}$ and $e \in X \cap Y$, then there exists a $Z \in \mathcal{C}$ such that $Z \subseteq \{X \cup Y\} - \{e\}$.

The 3 conditions above are referred to as *circuit axioms* .

In a matroid, the circuits (minimal dependent sets) are uniquely determined by the independent sets and vice versa. So the two definitions of matroid are equivalent.

1.5.2 Oriented matroids

Oriented matroids extend the concept of matroids by associating each element in the dependent set with a sign indicating its orientation, which is very useful in describing the relative positions of discrete geometric objects such as points, vectors and arrangements.

The following definition of oriented matroids is the extension of the second definition of matroids, which is in terms of circuits (minimal dependent sets).

Definition 1.5.3 *An Oriented matroid is denoted by $\mathcal{M} = (E, \mathcal{C})$, where*

E is a set of elements, each of which could be associated with a sign in $\{+, -\}$. A signed subset X of E shows how the elements in the subset is associated with the signs (an element that is not in the subset can be regarded as being associated with 0, i.e., without any sign.) We use X^+ to denote the elements in X associated with positive sign and X^- the elements associated with negative sign.

$\mathcal{C}$ is a set of circuits , i.e., a collection of signed subsets of E satisfying the following 4 properties (circuit axioms).

(C1) Empty set is not a circuit ($\phi \notin \mathcal{C}$).

(C2) The negative of circuit is a circuit ($X \in \mathcal{C} \Rightarrow -X \in \mathcal{C}$).

(C3) No proper subset of a circuit is a circuit ($X \in \mathcal{C}, Y \subset X \Rightarrow Y \notin \mathcal{C}$).

(C4) If $X, Y \in \mathcal{C}$ *with* $X \neq Y$ *and* $e \in X^+ \cap Y^-$, *then there is a third circuit* $Z \in \mathcal{C}$ *satisfying* $Z^+ \subseteq (X^+ \cup Y^+) \backslash \{e\}$ *and* $Z^- \subseteq (X^- \cup Y^-) \backslash \{e\}$.

For some $e \in E$ and $X \in \mathcal{C}$, we use X_e to denote the sign of e in X. X_e is also called *sign signature.*

The *vectors* of an oriented matroid (not of a vector space) are *composed* repeatedly from the circuits in the following way.

$$(X \circ Y)_e = \begin{cases} X_e & if X_e \neq 0, \\ Y_e & otherwise. \end{cases}$$

A straight-forward example of oriented matroids comes from the linear dependencies of a *vector configuration* $V = \{v_1, v_2, \cdots, v_n\} \subseteq I\!R^d$. Without ambiguity, let $V \in I\!R^{d \times n}$ also be the matrix of the n vectors. Then the space of linear dependencies is

$$Dep(V) := \{u \in I\!R^n \mid Vu = \mathbf{0}\}. \tag{1.5.2}$$

The vectors in the corresponding oriented matroid are the signed vectors in $Dep(V)$ and the circuits are the signed vectors of the minimal dependencies in $Dep(V)$.

Example 1 *For the following vector configuration (see Figure 1.1)*

$$V = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\},$$

the circuits are $\left\{ \begin{pmatrix} + \\ - \\ + \\ 0 \end{pmatrix}, \begin{pmatrix} + \\ - \\ 0 \\ + \end{pmatrix}, \begin{pmatrix} + \\ 0 \\ - \\ + \end{pmatrix}, \begin{pmatrix} 0 \\ + \\ - \\ + \end{pmatrix} \right\}$ *and the negations of them. The vectors (of the oriented matroid) consist of* $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$*, the circuits and the compositions of the circuits, i.e., dependencies that are not minimal. The non-minimal dependencies are* $\left\{ \begin{pmatrix} + \\ + \\ - \\ + \end{pmatrix}, \begin{pmatrix} + \\ - \\ + \\ + \end{pmatrix}, \begin{pmatrix} + \\ - \\ + \\ - \end{pmatrix}, \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix} \right\}$ *and their negations.*

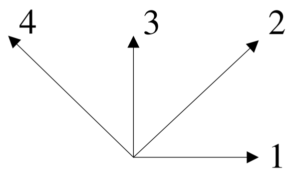

Figure 1.1: A vector configuration of 4 vectors

A linear arrangement corresponds naturally to an oriented matroid, as a linear arrangement is uniquely determined by the norm vectors of its hyperplanes, and the norm vectors form a vector configuration.

For more examples of combinatorial objects that fit into the model of oriented matroids, please refer to Appendix A.1. For a complete introduction to the rich theory of oriented matroids, please refer to [7].

1.5.3 Chirotopes

For a vector configuration $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ in $\mathbb{R}^r$, the *chirotope*, or *basis orientation*, is defined by the signs of the determinants of the ordered r-subset of the vectors, i.e., $\chi : \Lambda(n,r) \to \{+,-,0\}$ and $\chi(i_1,\dots,i_r) := \operatorname{sign}\det(\mathbf{v_{i_1}},\dots,\mathbf{v_{i_r}}) \in \{+,-,0\}$.

Example 2 *For a vector configuration* $\{\mathbf{v_1},\dots,\mathbf{v_n}\}$ *in* $I\!R^2$,

$$\begin{aligned}\chi(i,j) &= \operatorname{sign}\det[\mathbf{v_i},\mathbf{v_j}] \\ &= \operatorname{sign}\det\begin{pmatrix} \|\mathbf{v_i}\|\cos\theta_i & \|\mathbf{v_j}\|\cos\theta_j \\ \|\mathbf{v_i}\|\sin\theta_i & \|\mathbf{v_j}\|\sin\theta_j \end{pmatrix} \\ &= \operatorname{sign}\|\mathbf{v_i}\|\,\|\mathbf{v_j}\|\sin(\theta_j-\theta_i).\end{aligned}$$

So, $\chi(i,j)=0$ *if* $\mathbf{v_i}$ *and* $\mathbf{v_j}$ *are colinear;* $\chi(i,j)=+$ *if the angle distance from* $\mathbf{v_i}$ *to* $\mathbf{v_j}$ *is less than* π*, or* $\mathbf{v_i}$ *can be rotated counter-clockwise to the position of* $\mathbf{v_j}$ *by an angle less than* π*;* $\chi(i,j)=+$ *if the angle distance from* $\mathbf{v_i}$ *to* $\mathbf{v_j}$ *is bigger than* π*.*

The circuits of an oriented matroid determine an associated chirotope (unique up to a reversal of all the signs), and conversely, the circuits can be reconstructed from the chirotope.

An oriented matroid is *uniform* if its chirotope has no 0 in it. In the case of a vector configuration in $I\!R^r$, uniformness means that any r vectors in the configuration are linearly independent, i.e., there is no degeneracy.

We have the *3-term Grassmann-Plücker identity* :

$$\begin{array}{l}\det[\mathbf{v_1},\mathbf{v_2},\cdots,\mathbf{v_{r-2}},\mathbf{w_1},\mathbf{w_2}]\cdot\det[\mathbf{v_1},\mathbf{v_2},\cdots,\mathbf{v_{r-2}},\mathbf{w_3},\mathbf{w_4}]- \\ \det[\mathbf{v_1},\mathbf{v_2},\cdots,\mathbf{v_{r-2}},\mathbf{w_1},\mathbf{w_3}]\cdot\det[\mathbf{v_1},\mathbf{v_2},\cdots,\mathbf{v_{r-2}},\mathbf{w_2},\mathbf{w_4}]+ \\ \det[\mathbf{v_1},\mathbf{v_2},\cdots,\mathbf{v_{r-2}},\mathbf{w_1},\mathbf{w_4}]\cdot\det[\mathbf{v_1},\mathbf{v_2},\cdots,\mathbf{v_{r-2}},\mathbf{w_2},\mathbf{w_3}]=0\end{array} \tag{1.5.3}$$

for all $\mathbf{v_1},\mathbf{v_2},\cdots,\mathbf{v_{r-2}},\mathbf{w_1},\mathbf{w_2},\mathbf{w_3},\mathbf{w_4} \in I\!R^r$.

A chirotope is a representation of oriented matroids that is convenient for computational purposes due to its link to linear algebra. To reduce the number of signs involved in computations, *minimal reduced system* of a chirotope χ : $\Lambda(n, r) \to \{+, -, 0\}$ is introduced; it is defined as the minimal subset $\mathcal{S}$ of signs that completely determines χ, i.e., $\chi|_{\mathcal{S}} = \chi'|_{\mathcal{S}}$ together with the 3-term Grassmann-Plücker identity imply $\chi = \chi'$.

1.5.4 Realizable oriented matroids

There is a gap between the model of oriented matroids and geometric objects: although every vector configuration corresponds to an oriented matroid, not every oriented matroid corresponds to a vector configuration. The ones that do correspond are called *realizable*. Note that a couple of similar terms are used, including *stretchable* and *coordinatizable*. The formal definition follows.

Definition 1.5.4 *A realization of oriented matroid $\mathcal{M}$ of rank r on E is a mapping $\phi : E \to I\!R^r$ such that*

$$\chi(e_1, e_2, \cdots, e_r) = \operatorname{sign}\det(\phi(e_1), \phi(e_2), \cdots, \phi(e_r)) \tag{1.5.4}$$

for all $e_1, e_2, \cdots, e_r \in E$.

The realization problem, namely to determine whether a given oriented matroid is realizable, is known to be NP-hard [35]. One way to solve the problem is the so called method of *solvability sequences*; please refer to [6].

Remark 1.5.5 *It is known that an oriented matroid $\mathcal{M}$ of rank r on E is realizable if $r \leq 2$, or $r = 3$ and $|E| \leq 8$.*

1.6 Graphs

Polytopes and arrangements are related to graphs as the vertices and edges of a polytope or an arrangement correspond to the vertices and edges of a graph.

A graph is *d-polytopal* if it is the graph of some d-polytope. A *skeleton* in an arrangement is a connected subset of vertices and edges of the arrangement. In this thesis, we are particularly interested in the *envelope skeleton*, whose vertices and edges are external, i.e., belong to the envelope of the arrangement.

For a concise introduction to graph theory, please refer to [5]. We recall some results used in this thesis.

A *simple* graph is an unweighted, undirected graph containing no graph loops or multiple edges. A graph is *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A graph is said to be *k-connected* if there does not exist a set of $k-1$ vertices whose removal disconnects the graph.

Theorem 1.6.1 (Euler's formula) *Let f_0, f_1, f_2 be the number of vertices, edges and faces of a planar graph respectively, then*

$$f_0 - f_1 + f_2 = 2.$$

Theorem 1.6.2 (Steinitz' theorem) *A graph is 3-polytopal if and only if it is simple, planar, and 3-connected.*

Chapter 2

Introduction

2.1 Conjectured bound for the average diameter

Let $\delta(\mathcal{A})$ denote the average diameter of a bounded cell P_i of $\mathcal{A}$; that is,

$$\delta(\mathcal{A}) = \frac{\sum_{i=1}^{f_d^+(\mathcal{A})} \delta(P_i)}{f_d^+(\mathcal{A})}.$$

where $f_d^+(\mathcal{A})$ is the number of bounded cells of $\mathcal{A}$ and $\delta(P_i)$ denotes the diameter of P_i , i.e., the smallest number such that any two vertices of P_i can be connected by a path with at most $\delta(P_i)$ edges. Let $\Delta_{\mathcal{A}}(d, n)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by n inequalities in dimension d.

The main focus of this thesis is the following conjecture proposed in Deza, Terlaky and Zinchenko [12].

Conjecture 2.1.1 *The average diameter of a simple arrangement is bounded by its dimension from above, i.e.,* $\Delta_{\mathcal{A}}(d, n) \leq d$.

A simple line arrangement with average diameter equal to $2 - \frac{2}{n-1}$ was given in [12]. We propose, in Chapter 3, a line arrangement with average diameter $2-\frac{2\lceil\frac{n}{2}\rceil}{(n-1)(n-2)}$ and show that this diameter is maximal, i.e., $\Delta_{\mathcal{A}}(2,n) = 2-\frac{2\lceil\frac{n}{2}\rceil}{(n-1)(n-2)}$. In Chapter 4, a plane arrangement with average diameter $3-\frac{6}{n-1}+\frac{6(\lfloor\frac{n}{2}\rfloor-2)}{(n-1)(n-2)(n-3)}$ is proposed, yielding $\Delta_{\mathcal{A}}(3,n) \geq 3-\frac{6}{n-1}+\frac{6(\lfloor\frac{n}{2}\rfloor-2)}{(n-1)(n-2)(n-3)}$. In Section 5, the constructions in lower dimensions are generalized to general dimensions and we propose an hyperplane arrangement with $\binom{n-d}{d}$ cubical cells for $n \geq 2d$. It implies that the dimension d is an asymptotic lower bound for $\Delta_{\mathcal{A}}(d,n)$ for fixed d.

2.2 Hirsch Conjecture

Hirsch Conjecture was formulated in 1957 and reported in [10]. It states that the diameter of a polytope defined by n inequalities in dimension d is not greater than $n-d$.

Remark 2.2.1 *The conjecture does not hold for unbounded polyhedra [24].*

Deza, Terlaky and Zinchenko [12] noted the following link between the Hirsch conjecture and Conjecture 2.1.1.

Proposition 2.2.2 *If the conjecture of Hirsch holds for polytopes in dimension d, then $\Delta_{\mathcal{A}}(d,n) \leq d+\frac{2d}{n-1}$.*

Proof. Let $\{P_i \mid i = 1, 2, \cdots, f_d^+(\mathcal{A}_{d,n}) = \binom{n-1}{d}\}$ denote the set of bounded cells and $f_{d-1}(P_i)$ the number of facets of P_i. Then,

$$\begin{aligned}
\delta(\mathcal{A}_{d,n}) &= \frac{1}{f_d^+(\mathcal{A}_{d,n})} \sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} \delta(P_i) \\
&\le \frac{1}{f_d^+(\mathcal{A}_{d,n})} \sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} (f_{d-1}(P_i) - d) \qquad \text{(Hirsch conjecture)} \\
&= \frac{1}{f_d^+(\mathcal{A}_{d,n})} \sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} f_{d-1}(P_i) - d.
\end{aligned}$$

As a facet belongs to at most 2 bounded cells, $\sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} f_{d-1}(P_i)$ is at most twice the number of bounded facets in $\mathcal{A}_{d,n}$. Thus,

$$\begin{aligned}
\delta(\mathcal{A}_{d,n}) &\le \frac{2\left(f_{d-1}^+(\mathcal{A}_{d,n}) + f_{d-1}^0(\mathcal{A}_{d,n})\right)}{f_d^+(\mathcal{A}_{d,n})} - d \\
&= \frac{2n\binom{n-2}{d-1}}{\binom{n-1}{d}} - d \qquad \text{(Theorem 1.2.5, Lemma 1.2.6)} \\
&= d + \frac{2d}{n-1}.
\end{aligned}$$

□

The Hirsch conjecture holds for $d \le 3$ [23] and 0/1 polytopes [26]. Particularly, we have the following propositions:

Proposition 2.2.3 *Let P be a 2-polytope with n (non-redundant) facets. Then*

$$\delta(P) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Proposition 2.2.4 *Let P be a 3-polytope with n facets. Then*

$$\delta(P) \le \left\lfloor \frac{2n}{3} \right\rfloor - 1.$$

Proposition 2.2.3 is trivial and Proposition 2.2.4 is given in [21] (Chapter 16).

2.3 Conjectured bound for the envelope complexity

In the proof of Proposition 2.2.2, we noticed that

$$\sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} f_{d-1}(P_i) \leq 2\left(f_{d-1}^+(\mathcal{A}_{d,n}) + f_{d-1}^0(\mathcal{A}_{d,n})\right).$$

Precisely, as an external facet belongs to exactly one bounded facet, we have

$$\sum_{i=1}^{f_d^+(\mathcal{A}_{d,n})} f_{d-1}(P_i) = 2\left(f_{d-1}^+(\mathcal{A}_{d,n}) + f_{d-1}^0(\mathcal{A}_{d,n})\right) - f_{d-1}^0(\mathcal{A}_{d,n}). \tag{2.3.1}$$

Therefore, a lower bound for $f_{d-1}^0(\mathcal{A}_{d,n})$ would yield a tighter upper bound for $\Delta_{\mathcal{A}}(d,n)$. Let $\Phi_{\mathcal{A}}^0(d,n)$ be the minimum number of external facets for any simple arrangement defined by n hyperplanes in dimension d.

Conjecture 2.3.1 *Any arrangement $\mathcal{A}_{d,n}$ has at least $d\binom{n-1}{d-1}$ external facets, i.e., $\Phi_{\mathcal{A}}^0(d,n) \geq d\binom{n-1}{d-1}$.*

We show that $\Phi_{\mathcal{A}}^0(2,n) = 2(n-1)$ for $n \geq 4$ in Chapter 3, and that $\Phi_{\mathcal{A}}^0(3,n) \geq \frac{n(n-2)}{3} + 2$ for $n \geq 5$ in Chapter 4.

2.4 Haimovich bound for the expected number of pivot

While the complexity analysis on simplex methods is concerned with the number of pivot steps needed to reach the optimum, the bounds of the average diameter of arrangements could also provide some insights into the average complexity of simplex methods. Haimovich's probabilistic analysis on the shadow-vertex algorithm shows that the expected number of pivot steps needed in

Phase II is bounded by d, the dimension, which is also the bound on the average diameter in Conjecture 2.1.1, see Section 0.7 of [3] for more details.

While Haimovich's results and Conjecture 2.1.1 have a similar flavor, they differ in some aspects: Haimovich considers unbounded cells in addition to the bounded ones, and the number of pivots could be smaller than the diameter for some cells.

2.5 Dedieu-Malajovich-Shub bound for the average curvature

Intuitively, the *curvature* is a measure of how far a geometric object deviates from being flat. In particular, we are interested in the curvature of the central path of a polytope defined by n inequalities in dimension d. For a compact introduction to the theory of interior point methods, as well as the concept of central path, please refer to [32, 33]. A detailed description of the curvature of curves can be found in Chapter 17 of [8].

Let $\mathcal{C}$ be a curve of length L in $I\!R^d$ and $\psi_{arc} : [0, L] \to I\!R^d$ its parametrization by the curve length. Then the *curvature* at the point t is $\kappa(t) = \ddot{\psi}_{arc}(t)$ and the *total curvature* is defined as $\int_0^L \|\kappa(t)\| dt$.

Proposition 2.5.1 *The average total curvature of a bounded cell of a simple arrangement determined by n inequalities in dimension d is not greater than $2\pi d$ [11].*

Conjecture 2.1.1 is clearly a discrete version of Proposition 2.5.1.

Chapter 3

Line Arrangements with Maximal Average Diameter

3.1 Line arrangements with large average diameter

For $n \geq 4$, we consider the simple line arrangement $\mathcal{A}^o_{2,n}$ made of the 2 lines h_1 and h_2 forming, respectively, the x_1 and x_2 axis, and $(n-2)$ lines defined by their intersections with h_1 and h_2. We have $h_k \cap h_1 = \{1 + (k-3)\varepsilon, 0\}$ and $h_k \cap h_2 = \{0, 1-(k-3)\varepsilon\}$ for $k = 3, 4, \ldots, n-1$, and $h_n \cap h_1 = \{2, 0\}$ and $h_n \cap h_1 = \{0, 2+\varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < \frac{1}{n-3}$. See Figure 3.1 for an arrangement combinatorially equivalent to $\mathcal{A}^o_{2,7}$.

Proposition 3.1.1 *For $n \geq 4$, the bounded cells of the arrangement $\mathcal{A}^o_{2,n}$ consist of $(n-2)$ triangles, $\frac{(n-1)(n-4)}{2}$ 4-gons, and one n-gon.*

Proof. The first $(n-1)$ lines of $\mathcal{A}^o_{2,n}$ clearly form a simple line arrangement which bounded cells are $(n-3)$ triangles and $\binom{n-3}{2}$ 4-gons. The last line h_n adds one n-gons, one triangle and $(n-4)$ 4-gons. $\square$

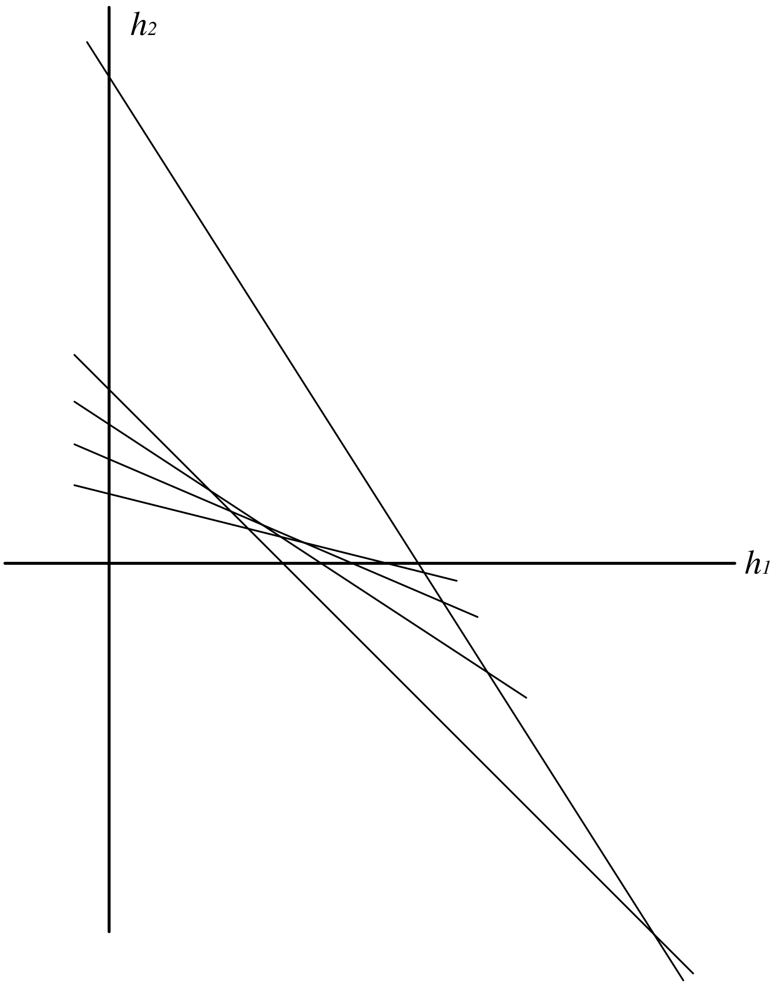

Figure 3.1: An arrangement combinatorially equivalent to $\mathcal{A}^{o}_{2,7}$.

Corollary 3.1.2 *We have* $\delta(\mathcal{A}^o_{2,n}) = 2 - \frac{2\lceil \frac{n}{2}\rceil}{(n-1)(n-2)}$ *for* $n \geq 4$.

Proof. Since the diameter of a k-gons is $\lfloor \frac{k}{2} \rfloor$, we have $\delta(\mathcal{A}^o_{2,n}) = 2 - 2\frac{(n-2)-(\lfloor \frac{n}{2}\rfloor -2)}{(n-1)(n-2)} = 2 - \frac{2\lceil \frac{n}{2}\rceil}{(n-1)(n-2)}$. □

The first $(n-1)$ lines of $\mathcal{A}^o_{2,n}$ form the line arrangement $\mathcal{A}^*_{2,n-1}$ proposed in [12]. The arrangement $\mathcal{A}^*_{2,n}$ has $3(n-2)$ external facets and average diameter $\delta(\mathcal{A}^*_{2,n}) = 2 - \frac{2}{n-1}$. The arrangement $\mathcal{A}^o_{2,n}$ has $2(n-1)$ external facets and minimizes the number of external facets, see Theorem 3.2.1. Note that the envelope of the bounded cells of $\mathcal{A}^o_{2,n}$ has one reflex vertex. In Section 3.2 we show that the arrangement $\mathcal{A}^o_{2,n}$ is the best possible; that is, $\Delta_{\mathcal{A}}(2,n) = \delta(\mathcal{A}^o_{2,n}) = 2 - \frac{2\lceil \frac{n}{2}\rceil}{(n-1)(n-2)}$ for $n \geq 4$. In Section 4, following the same approach, we generalize $\mathcal{A}^*_{2,n-1}$ to dimension 3 and add one plane to reduce the number of external facets.

3.2 Exact value of the maximum average diameter in the plane

Clearly the number of external facets $f^0_1(\mathcal{A}_{2,n})$ is the same as the number of external vertices $f^0_0(\mathcal{A}_{2,n})$. The external vertices can be divided into three types, namely V_2, V_3 and V_4, being incident to, respectively, 2, 3, and 4 bounded edge. We use v_2, v_3 and v_4 to denote, respectively, the number of vertices of type V_2, V_3 and V_4. The number of external facets $f^0_1(\mathcal{A}_{2,n}) = v_2 + v_3 + v_4$ and we have the following property [4].

Proposition 3.2.1 *Any simple line arrangement has at least* $2(n-1)$ *external facets. Moreover,* $\Phi^0_{\mathcal{A}}(2,n) = 2(n-1)$.

Proof. Let us count the number of external vertices. Give each external vertex a weight of 1, then distribute the weight to the 2 lines intersecting at the vertex. Figure 3.2 shows how the weights are distributed for different types of external vertices. Now let us count the weights line-wise. For each line of the arrangement, there are two vertices from which the line is entering or leaving the envelope; we call them end points. Since V_4 cannot be an end point of a line, the possible combinations of the two end points are (V_2, V_2), (V_2, V_3) and (V_3, V_3) and the weights of the corresponding lines are at least 1, 2 and 2 respectively. The weight lower bound of the line L_{23} with ends (V_2, V_3) is not as straightforward as the other two. The ends (V_2, V_3) gives L_{23} a weight of $\frac{1}{2} + 1 = \frac{3}{2}$. A further look reveals that there has to be a V_4 along L_{23}, which gives L_{23} an extra weight of $\frac{1}{2}$. Therefore, the weight of L_{23} is at least 2. The lines L_{22}'s with ends (V_2, V_2) are the only ones with a weight that is less than 2. However, there can only be at most 2 of them, otherwise, any 3 such lines would force the envelope to be a triangle (2D simplex), which is impossible because it is known that the envelope of $\mathcal{A}_{2,n}$ can not be convex [15]. So the total weight, or the number of external vertices, counted is at least $2n - 2$. In $\mathcal{A}_{2,n}$ the number of external facets is equal to the number of external vertices. Therefore $\Phi^0_{\mathcal{A}}(2, n) \geq 2n - 2$ and since $f^0_1(\mathcal{A}^o_{2,n}) = 2(n - 1)$ (see Figure 3.1), we have $\Phi^0_{\mathcal{A}}(2, n) = 2(n - 1)$. $\square$

Theorem 3.2.2 *The arrangement $\mathcal{A}^o_{2,n}$ maximize the average diameter, i.e., $\Delta_{\mathcal{A}}(2, n) = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$ for $n \geq 3$.*

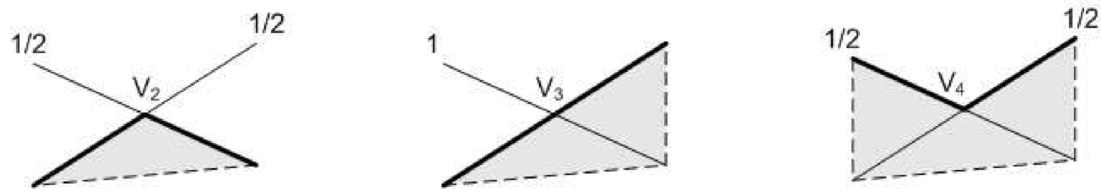

Figure 3.2: Weight distribution rules (the shaded area is inside the envelope).

Proof. Since

$$\begin{aligned}
\delta(\mathcal{A}_{2,n}) &= \frac{1}{f_2^+(\mathcal{A}_{2,n})} \sum_{i=1}^{f_2^+(\mathcal{A}_{2,n})} \delta(P_i) \\
&= \frac{1}{f_2^+(\mathcal{A}_{2,n})} \sum_{i=1}^{f_2^+(\mathcal{A}_{2,n})} \left\lfloor \frac{f_1(P_i)}{2} \right\rfloor && \text{(Proposition 2.2.3)} \\
&= \frac{1}{f_2^+(\mathcal{A}_{2,n})} \frac{\sum_{i=1}^{f_2^+(\mathcal{A}_{2,n})} f_1(P_i) - p_{odd}(\mathcal{A}_{2,n})}{2} \\
&= \frac{1}{f_2^+(\mathcal{A}_{2,n})} \frac{2n(n-2) - f_1^0(\mathcal{A}_{2,n}) - p_{odd}(\mathcal{A}_{2,n})}{2} && \text{(by 2.3.1)}
\end{aligned}$$

where $p_{odd}(\mathcal{A}_{2,n})$ is the number of odd-gons in $\mathcal{A}_{2,n}$. To maximize $\delta(\mathcal{A}_{2,n})$ is therefore equivalent to minimize $f_1^0(\mathcal{A}_{2,n}) + p_{odd}(\mathcal{A}_{2,n})$. By Theorem 3.2.1 and Lemma 1.2.9, we have $f_1^0(\mathcal{A}_{2,n}) \geq 2n-2$ and $p_{odd}(\mathcal{A}_{2,n}) \geq n-2$. Since both are satisfied with equality for $\mathcal{A}^o_{2,n}$ for even n, we have $\Delta_{\mathcal{A}}(2,n) = \delta(\mathcal{A}^o_{2,n})$ for even n. For odd n, $\mathcal{A}_{2,n}$ has $n-1$ odd gons and the only way to improve $\delta(\mathcal{A}^o_{2,n})$ would by having one less odd gon, which is impossible as otherwise all the odd gons are the $n-2$ triangles and hence $\sum_{i=1}^{f_2^+(\mathcal{A}_{2,n})} f_1(P_i)$ is odd, while $\sum_{i=1}^{f_2^+(\mathcal{A}_{2,n})} f_1(P_i) = f_1(\mathcal{A}_{2,n}) - f_1^0(\mathcal{A}_{2,n}) = 2n(n-2) - 2(n-1)$ indicates that $\sum_{i=1}^{f_2^+(\mathcal{A}_{2,n})} f_1(P_i)$ is even. $\square$

Corollary 3.2.3 *Conjecture 2.1.1 holds for line arrangements.*

3.3 Line arrangements with fewer than 8 lines

There is only one combinatorial type of simple line arrangement for $n = 4$. For $n = 5$, there are 6 combinatorial types of simple line arrangement; see Figure 3.3.

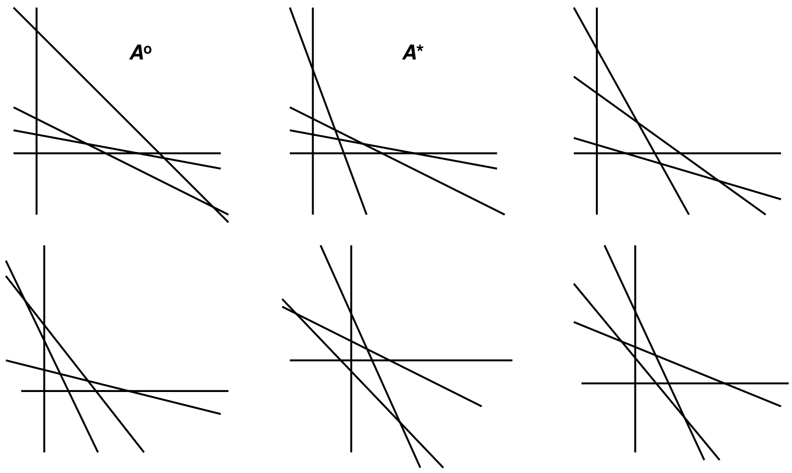

Figure 3.3: Enumeration of $\mathcal{A}_{2,5}$.

See Figure 3.4 for the visual enumeration [1] of all the 43 combinatorial types of simple arrangements formed by 6 lines. The arrangement $\mathcal{A}^0_{2,6}$ has the maximal average diameter [2]. For detailed computational results, please refer to Appendix C.1.

We analyzed 886 out of 922 combinatorial types for $\mathcal{A}_{2,7}$. The computational results are given in Appendix C.2.

[1] The visualization is realized using Maple 10.
[2] $\mathcal{A}^0_{2,6}$ is in the 6th row and 6th column of Figure 3.4.

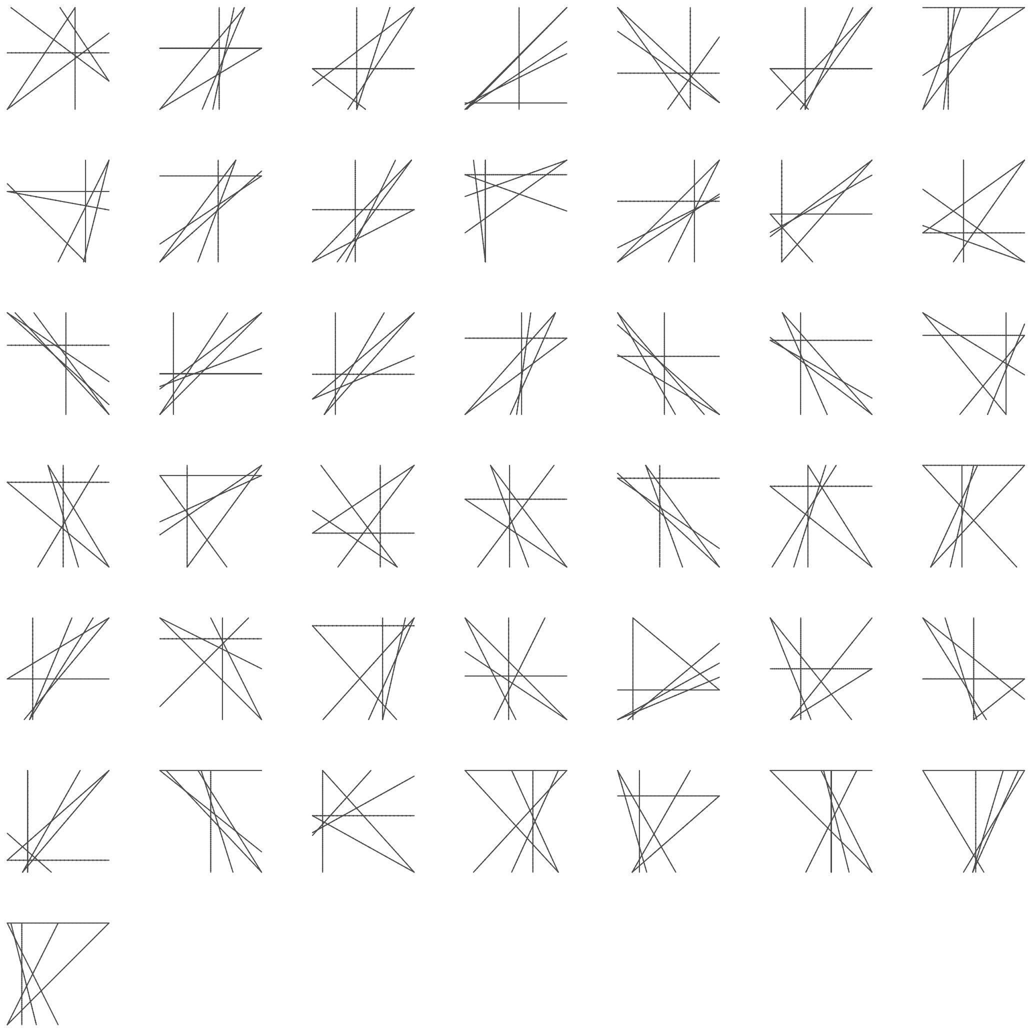

Figure 3.4: Enumeration of $\mathcal{A}_{2,6}$.

Chapter 4

Plane Arrangements with Large Average Diameter

4.1 Plane arrangements with large average diameter

For $n \geq 5$, we consider the simple plane arrangement $\mathcal{A}^o_{3,n}$ made of the 3 plane h_1, h_2 and h_3 corresponding, respectively, to $x_3 = 0$, $x_2 = 0$ and $x_1 = 0$, and $(n-3)$ planes defined by their intersections with the x_1, x_2 and x_3 axis. We have $h_k \cap h_1 \cap h_2 = \{1 + 2(k-4)\varepsilon, 0, 0\}$, $h_k \cap h_1 \cap h_3 = \{0, 1 + (k-4)\varepsilon, 0\}$ and $h_k \cap h_2 \cap h_3 = \{0, 0, 1-(k-4)\varepsilon\}$ for $k = 4, 5, \ldots, n-1$, and $h_n \cap h_1 \cap h_2 = \{3, 0, 0\}$, $h_n \cap h_1 \cap h_3 = \{0, 2, 0\}$ and $h_n \cap h_2 \cap h_3 = \{0, 0, 3 + \varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < \frac{1}{n-4}$. See Figure 4.1 for an illustration of an arrangement combinatorially equivalent to $\mathcal{A}^o_{3,7}$ where, for clarity, only the bounded cells belonging to the positive orthant are drawn.

Proposition 4.1.1 *For $n \geq 5$, the bounded cells of the arrangement $\mathcal{A}^o_{3,n}$ consist of $(n-3)$ tetrahedra, $(n-3)(n-4)-1$ cells combinatorially equivalent to a prism with a triangular base, $\binom{n-3}{3}$ cells combinatorially equivalent to a*

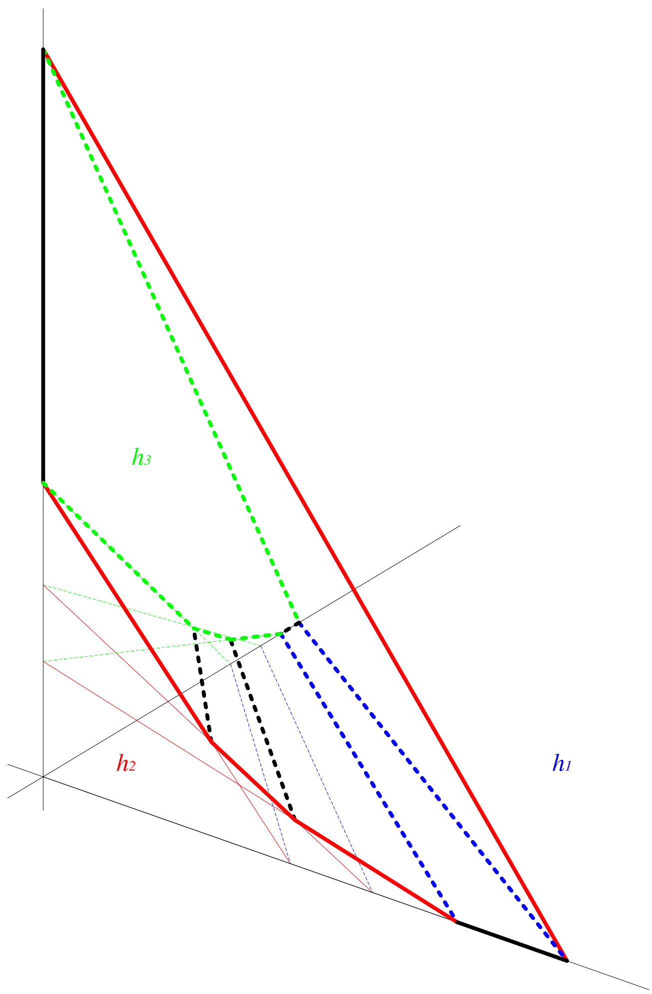

Figure 4.1: An arrangement combinatorially equivalent to $\mathcal{A}^o_{3,7}$.

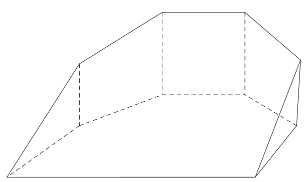

Figure 4.2: A polytope combinatorially equivalent to the shell S_7.

cube, and one cell combinatorially equivalent to a shell S_n with n facets and $2(n-2)$ vertices. See Figure 4.2 for an illustration of S_7.

Proof. For $4 \leq k \leq n-1$, let $\mathcal{A}^*_{3,k}$ denote the arrangement formed by the first k planes of $\mathcal{A}^o_{3,n}$. See Figure 4.3 for an arrangement combinatorially equivalent to $\mathcal{A}^*_{3,6}$. We first show by induction that the bounded cells of the arrangement $\mathcal{A}^*_{3,n-1}$ consist of $(n-4)$ tetrahedra, $(n-4)(n-5)$ combinatorial triangular prisms and $\binom{n-4}{3}$ combinatorial cubes. We use the following notation to describe the bounded cells of $\mathcal{A}^*_{3,k-1}$: $T_\triangle$ for a tetrahedron with a facet on h_1 ; $P_\triangle$, respectively $P_\diamond$, for a combinatorial triangular prism with a triangular, respectively square, facet on h_1; $C_\diamond$ for a combinatorial cube with a square facet on h_1; and C, respectively T and P, for a combinatorial cube, respectively tetrahedron and triangular prism, not touching h_1. When the plane h_k is added, the cells $T_\triangle$, $P_\triangle$, $P_\diamond$, and $C_\diamond$ are sliced, respectively, into T and $P_\triangle$, P and $P_\triangle$, P and $C_\diamond$, and C and $C_\diamond$. In addition, one $T_\triangle$ cell and $(k-4)$ $P_\diamond$ cells are created by bounding $(k-3)$ unbounded cells of $\mathcal{A}^*_{3,k-1}$. Let $c(k)$ denotes the number of C cells of $\mathcal{A}^*_{3,k}$, similarly for $C_\diamond$, T, $T_\triangle$, P, $P_\triangle$ and $P_\diamond$. For $\mathcal{A}^*_{3,4}$ we have $t_\triangle(4) = 1$ and $t(4) = p(4) = p_\triangle(4) = p_\diamond(4) = c(4) = c(4) = 0$. The addition of h_k removes and adds one $T_\triangle$, thus, $t_\triangle(k) = 1$. Similarly, all $P_\diamond$ are removed and $(k-4)$ are added, thus, $p_\diamond(k) = (k-4)$. Since $t(k) = t(k-1) + t_\triangle(k-1)$ and $p_\triangle(k) = p_\triangle(k-1) + t_\triangle(k-1)$, we have $t(k) = p_\triangle(k) = (k-4)$. Since $p(k) = p(k-1) + p_\triangle(k-1) + p_\diamond(k-1)$, we have $p(k) = (k-4)(k-5)$. Since $c_\diamond(k) = c_\diamond(k-1) + p_\diamond(k-1)$, we have $c_\diamond(k) = \binom{k-4}{2}$. Since $c(k) = c(k-1)+c_\diamond(k-1)$, we have $c(k) = \binom{k-4}{3}$. Therefore the bounded cells of $\mathcal{A}^*_{3,n-1}$ consist of $t(n-1)+t_\triangle(n-1) = (n-4)$ tetrahedra, $p(n-1) + p_\triangle(n-1) + p_\diamond(n-1) = (n-4)(n-5)$ combinatorial triangular prisms, and $c(n-1)+c_\diamond(n-1)=\binom{n-4}{3}$ combinatorial cubes. The addition of h_n to $\mathcal{A}^*_{3,n-1}$ creates 1 shell S_n with 2 triangular facets belonging to h_2 and h_3 and 1 square facet belonging to h_1. Besides S_n, all the bounded cells created by the addition of h_n are below h_1. One $P_\diamond$ and $n-5$ combinatorial cubes are created between h_2 and h_3. The other bounded cells are on the negative side of h_3: $n-5$ $P_\diamond$ and 1 $T_\triangle$ between h_n and h_{n-1}, and $n-k-5$ $C_\diamond$ and 1 $P_\triangle$ between h_{n-k} and h_{n-k-1} for $k=1,\dots,n-5$. In total, we have 1 tetrahedron, $\binom{n-4}{2}$ combinatorial cubes and $(2n-9)$ combinatorial triangular prisms below h_1. □

Corollary 4.1.2 *We have* $\delta(\mathcal{A}^o_{3,n})=3-\frac{6}{n-1}+\frac{6(\lfloor\frac{n}{2}\rfloor-2)}{(n-1)(n-2)(n-3)}$ *for* $n\geq 5$.

Proof. Since the diameter of a tetrahedron, triangular prism, cube and n-shell is, respectively, $1,2,3$ and $\lfloor\frac{n}{2}\rfloor$, we have

$$\begin{aligned}\delta(\mathcal{A}^o_{3,n}) &= 3-6\frac{2(n-3)+(n-3)(n-4)-1-(\lfloor\frac{n}{2}\rfloor-3)}{(n-1)(n-2)(n-3)}\\ &= 3-\frac{6}{n-1}+\frac{6(\lfloor\frac{n}{2}\rfloor-2)}{(n-1)(n-2)(n-3)}.\end{aligned}$$

□

4.2 Improved upper bound for plane arrangements

The dimension 3 case is slightly more complicated than the plane case. As the union of all of the bounded cells is a piecewise linear ball, see [13], the envelope is planar for plane arrangements. By Theorem 1.6.1 (Euler's formula), we have

$$f_0^0(\mathcal{A}_{3,n})-f_1^0(\mathcal{A}_{3,n})+f_2^0(\mathcal{A}_{3,n})=2. \tag{4.2.1}$$

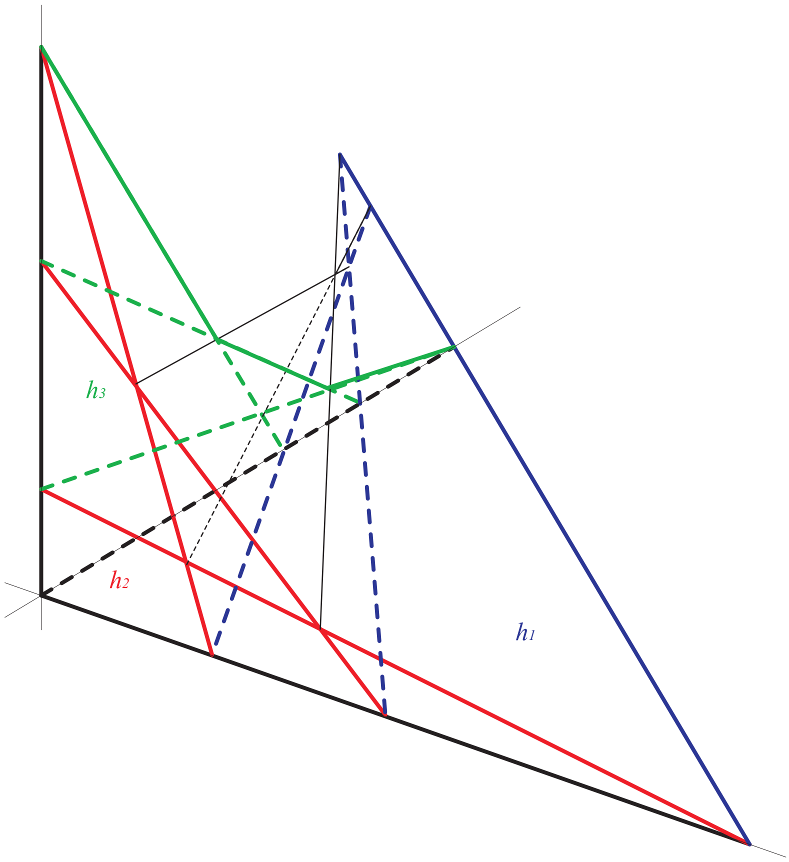

Figure 4.3: An arrangement combinatorially equivalent to $\mathcal{A}^*_{3,6}$.

$$2f_1^0(\mathcal{A}_{3,n}) \geq 3f_0^0(\mathcal{A}_{3,n}). \tag{4.2.2}$$

The external vertices of $\mathcal{A}_{3,n} \cap h_i$ are external vertices of $\mathcal{A}_{3,n}$ for $i = 1, 2, \ldots, n$. $\mathcal{A}_{3,n} \cap h_i$ has at least $2(n-2)$ external facets (Theorem 1.2.6), i.e., at least $2(n-2)$ external vertices. Since a vertex belongs to 3 planes, it is counted three times and, therefore, $\mathcal{A}_{3,n}$ has at least $\frac{2n(n-2)}{3}$ external vertices; that is:

$$f_0^0(\mathcal{A}_{3,n}) \geq \frac{2n(n-2)}{3}. \tag{4.2.3}$$

Lemma 4.2.1 *Any simple plane arrangement has at least* $\frac{n(n-2)}{3} + 2$ *external facets, i.e.,* $\Phi_{\mathcal{A}}^0(3,n) \geq \frac{n(n-2)}{3} + 2$.

Proof.

$$\begin{aligned}
f_2^0(\mathcal{A}_{3,n}) &= f_1^0(\mathcal{A}_{3,n}) - f_0^0(\mathcal{A}_{3,n}) + 2 && \text{(by 4.2.1)}\\
&\geq \frac{f_0^0(\mathcal{A}_{3,n})}{2} + 2 && \text{(by 4.2.2)}\\
&\geq \frac{n(n-2)}{3} + 2. && \text{(by 4.2.3)}
\end{aligned}$$

□

Using Lemma 4.2.1 and the fact that the Hirsch conjecture holds for $d = 3$, we get the following upper bound for the average diameter.

Proposition 4.2.2 *We have* $\Delta_{\mathcal{A}}(3,n) \leq 3 + \frac{4(2n^2-16n+21)}{3(n-1)(n-2)(n-3)}$.

Proof. We have:

$$
\begin{aligned}
\delta(\mathcal{A}) &= \frac{1}{f_3^+(\mathcal{A}_{3,n})} \sum_{i=1}^{f_3^+(\mathcal{A}_{3,n})} \delta(P_i) \\
&\leq \frac{1}{f_3^+(\mathcal{A}_{3,n})} \sum_{i=1}^{f_3^+(\mathcal{A}_{3,n})} \left(\left\lfloor \frac{2}{3} f_2(P_i) \right\rfloor - 1 \right) && \text{(Proposition 2.2.4)} \\
&= \frac{1}{f_3^+(\mathcal{A}_{3,n})} \sum_{i=1}^{f_3^+(\mathcal{A}_{3,n})} \left\lfloor \frac{2}{3} f_2(P_i) \right\rfloor - 1 \\
&\leq \frac{1}{f_3^+(\mathcal{A}_{3,n})} \cdot \left(\sum_{i=1}^{f_3^+(\mathcal{A}_{3,n})} \frac{2}{3} f_2(P_i) - \frac{2}{3}(n-3) \right) - 1 && \text{(Corollary 1.2.9)} \\
&= \frac{1}{f_3^+(\mathcal{A}_{3,n})} \cdot \frac{2}{3} \left(\sum_{i=1}^{f_3^+(\mathcal{A}_{3,n})} f_2(P_i) - (n-3) \right) - 1 \\
&\leq \frac{1}{f_3^+(\mathcal{A}_{3,n})} \cdot \frac{2}{3} \left(2n \binom{n-2}{2} - \left(\frac{n(n-2)}{3} + 2 \right) - (n-3) \right) - 1 \\
& && \text{(Lemma 1.2.6, 4.2.1)} \\
&= 3 + \frac{4(2n^2 - 16n + 21)}{3(n-1)(n-2)(n-3)}.
\end{aligned}
$$

□

4.3 Plane arrangements with fewer than 7 planes

As there is only one combinatorial type of simple plane arrangement for $n = 5$, we have $\Delta_{\mathcal{A}}(3, 5) = \delta(\mathcal{A}^o_{3,5}) = \frac{3}{2}$.

While the diameter of $\mathcal{A}^o_{3,n}$ is arbitrarily close to 3 as n goes to infinity, we do not believe it has the maximal average diameter. Among the 43 simple combinatorial types of arrangements formed by 6 planes, the maximum average diameter is 2 while the $\delta(\mathcal{A}^o_{3,6}) = 1.8$. One of the 2 simple arrangements with

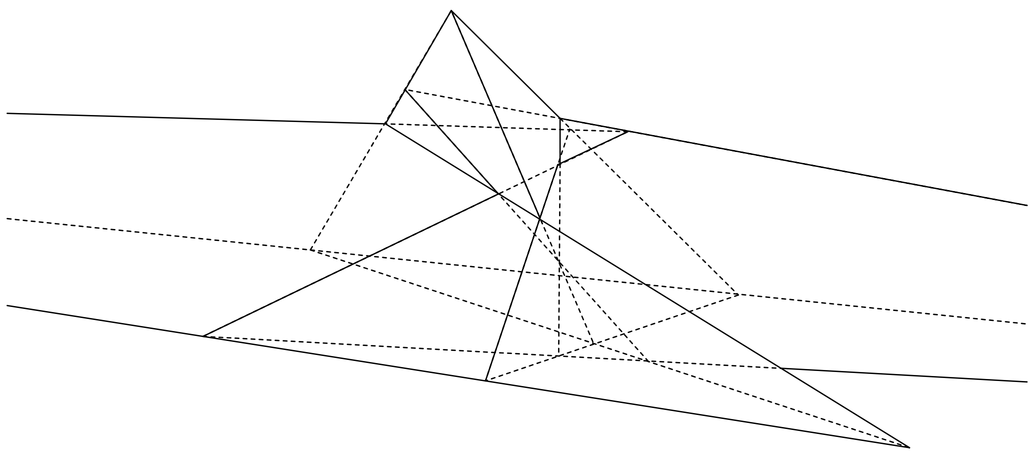

Figure 4.4: An arrangement maximizing the average diameter for $(d, n) = (3, 6)$.

maximum diameter is shown in Figure 4.4[1]. It has 3 simplices, 4 simplex prisms, and 3 6-shells. For detailed computational results, see Appendix C.3.

[1]The visualization is realized using Maple 10.

Chapter 5

Hyperplane Arrangements with Large Average Diameter

In Section 5.2, the arrangements $\mathcal{A}^*_{2,n}$ and $\mathcal{A}^*_{3,n}$ are generalized to an hyperplane arrangement $\mathcal{A}^*_{d,n}$ which contains $\binom{n-d}{d}$ cubical cells for $n \geq 2d$. It implies that the average diameter $\delta(\mathcal{A}^*_{d,n})$ is arbitrarily close to d for n large enough. Thus, the dimension d is an asymptotic lower bound for $\Delta_{\mathcal{A}}(d,n)$ for fixed d. Before presenting in Section 5.2 the arrangement $\mathcal{A}^*_{d,n}$, we recall in Section 5.1 the combinatorial structure of a simple arrangement formed by $d+2$ hyperplanes in dimension d.

5.1 Simple arrangement with $d+2$ hyperplanes

We consider a d-dimensional simple arrangement $\mathcal{A}_{d,d+2}$ formed by $d+2$ hyperplanes. We present two proofs of the uniqueness of the combinatorial type of $\mathcal{A}_{d,d+2}$; one more geometrical and one using Gale transform. The combinatorial structure of $\mathcal{A}_{d,d+2}$ is also given.

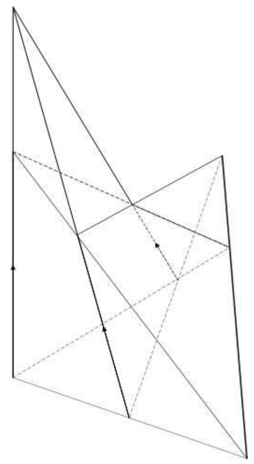

Figure 5.1: An arrangement combinatorially equivalent to $\mathcal{A}_{3,5}$.

5.1.1 Uniqueness of $\mathcal{A}_{d,d+2}$ – geometrical approach

Proposition 5.1.1 *Let P a simple d-dimensional polytope with $d+2$ facets. If P has a simplex facet, then P is combinatorially equivalent to a prism with a simplex basis.*

Proof. Suppose that polytope P is formed by the set of hyperplanes $H = \{h_0, h_1, \ldots, h_{d+1}\}$ and the known simplex facet is on h_0. Let the d vertices of the simple facet be $v_1, v_2, \ldots, v_d$ and $v_i (1 \leqslant i \leqslant d)$ is the intersection of $H \setminus \{h_i, h_{d+1}\}$. Since P is simple, each v_i is adjacent to exactly one other vertex v'_i that is not on h_0. because the sets of hyperplanes that determine the adjacent vertices v_i and v'_i differ by only one hyperplane, v'_i can only be the intersection of $H \setminus \{h_0, h_i\}$ or $H \setminus \{h_0, h_{d+1}\}$. Note that v'_i can not be the intersection of $H \setminus \{h_0, h_{d+1}\}$, otherwise $H \setminus \{h_{d+1}\}$ forms a simplex, which is impossible. Therefore, v'_i is the intersection of $H \setminus \{h_0, h_i\}$, hence v'_i is on h_{d+1} for all i. $v_1, \ldots, v_d, v'_1, \ldots, v'_d$ are all the vertices of P and $v_1, \ldots, v_d \in$

$h_0, v_1', \ldots, v_d' \in h_{d+1}$. So P is combinatorially equivalent to a prism with a simplex basis. $\square$

Let us call a polytope combinatorially equivalent to a prism with a simplex basis a *simplex prism*, and similarly for cubes. As showed later, the majority of the bounded cells in $\mathcal{A}_{d,d+2}$ has $d+2$ facets. The above proposition provides a tool to identify a simplex prism cell.

Proposition 5.1.2 *$\mathcal{A}_{d,d+2}$ is combinatorially unique.*

Proof. We prove it by induction on d. It is true for $d = 2$, and suppose that $\mathcal{A}_{d',d'+2}$ is combinatorially unique for $d' < d$. Let $\mathcal{A}_{d,d+2}$ be formed by hyperplanes $h_1, h_2, \ldots, h_{d+2}$. By Corollary 1.2.9, it has a simplex cell. Without losing generality, we assume that the simplex cell is tightly bounded by $h_1, \ldots, h_{d+1}$. h_{d+2} will not pass through the simplex cell, thus gives rise to a simplex prism. Let the two basis of the prism be on h_{d+1} and h_{d+2}, which intersect at a $(d-2)$-face F. All the bounded cells of $\mathcal{A}_{d,d+2}$ are between h_{d+1} and h_{d+2} except the simplex cell (see Figure 5.2 for an example in 3D). The projection of $\mathcal{A}_{d,d+2}$ on h_{d+2} is $\mathcal{A}_{d-1,d+1}$. The bounded cells between h_{d+1} and h_{d+2} can be viewed as formed by raising $\mathcal{A}_{d-1,d+1}$ about axis F. By induction hypothesis $\mathcal{A}_{d-1,d+1}$ is combinatorially unique. So $\mathcal{A}_{d,d+2}$ is also combinatorially unique. $\square$

5.1.2 Uniqueness of $\mathcal{A}_{d,d+2}$ – Gale transform approach

A point configuration P of n points $\{\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}\}$ in $I\!R^d$ corresponds to a vector configuration V of n vectors $\left\{ \begin{bmatrix} \mathbf{x_1} \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{x_2} \\ 1 \end{bmatrix}, \cdots, \begin{bmatrix} \mathbf{x_n} \\ 1 \end{bmatrix} \right\}$ in $I\!R^{d+1}$. They can be mapped to a vector configuration V^* of n vectors in $I\!R^{n-d-1}$ through *Gale transform*, and further more, to a signed point configuration P^*

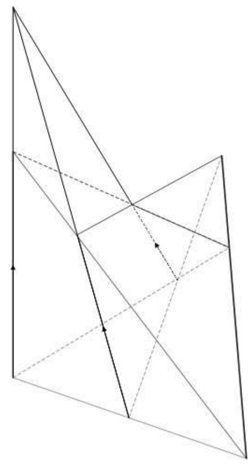

Figure 5.2: Getting $\mathcal{A}_{3,5}$ by dimension lifting - the bounded cells between the green and red plane are lifted from $\mathcal{A}_{2,4}$ about the blue line.

in $I\!R^{n-d-2}$ through *affine Gale transform.* Gale transform preserves combinatorial properties, i.e., P, V, V^* and P^* all have the same oriented matroid[1], making it useful for studying point/vector configurations having only few more points/vectors than the dimension as it drastically decreases the dimension of the geometric object studied. An introduction to the theoretical background and implementing issues of Gale transform can be found in [38](Chapter 6) and [1], respectively.

A proof of Proposition 5.1.2 using Gale transform follows.

Proof. $\mathcal{A}_{d,d+2}$ corresponds to a linear arrangement in $I\!R^{d+1}$ through the technique of dimension lifting (see Figure 5.3). An extra linear hyperplane h_{d+3} is needed to record the direction of dimension lifting. Without losing generality, let h_{d+3} be $x_{d+1} = 1$, whose normal vector is $\mathbf{e_{d+3}}$. The norms of the

[1]The oriented matroids associated to P (or V) and V^* (or P^*) are dual to each other (oriented matroid duality is not formally introduced in this thesis, see [7].

hyperplanes of the linear arrangement give rise to a vector configuration V of $d+3$ vectors $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_{d+2}}, \mathbf{e_{d+3}}\}$. Through Gale transform, V can be mapped to a vector configuration V^* of $d+3$ vectors in $I\!R^2$, and furthermore, to a point configuration P^* of $d+3$ distinct signed points $\{\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_{d+2}}, \mathbf{x_{d+3}}\}$ in $I\!R$ (a line). Notice that $\mathbf{x_{d+3}}$ is fixed for all arrangements. Since the arrangement is simple, any point in P^* is either black(positive) or white(negative). For a point configuration like that, only through relabeling and/or recoloring (of the points other than $\mathbf{x_{d+3}}$) can we get different oriented matroids. However they are all associated with the same arrangement combinatorially, because the relabeling of the points corresponds to the relabeling of the hyperplanes in $\mathcal{A}_{d,d+2}$; and the recoloring of the points corresponds to the reorientation of the hyperplanes, i.e., switching the positive and negative sides of the hyperplanes, which does not affect the combinatorial structure of the arrangement. Therefore, $\mathcal{A}_{d,d+2}$ is combinatorially unique. $\square$

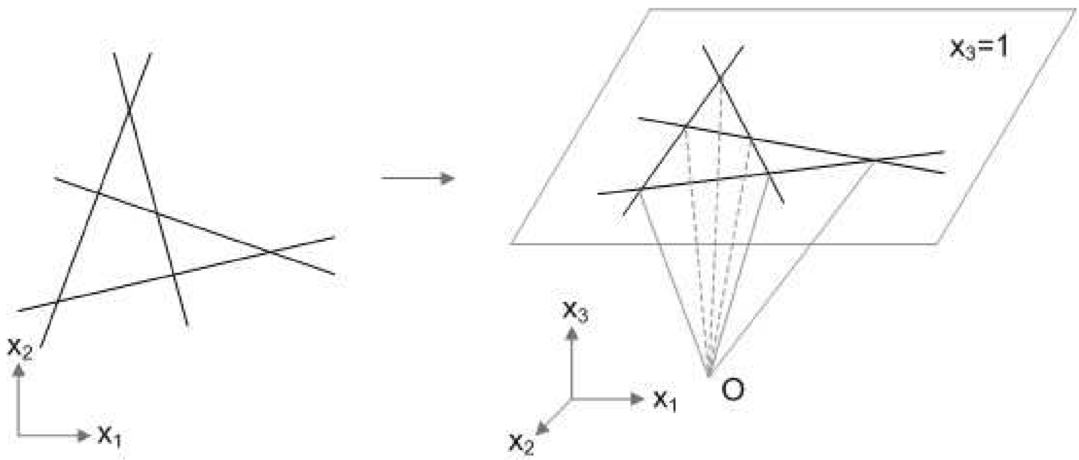

Figure 5.3: Lifting an arrangement from dimension 2 to dimension 3.

5.1.3 Combinatorial structure of $\mathcal{A}_{d,d+2}$

Theorem 1.2.5 gives that $\mathcal{A}_{d,d+2}$ has $\binom{d+1}{d} = d+1$ bounded cells. Obviously, all the bounded cells have either $d+1$ or $d+2$ facets. The dimension lifting process discussed above actually gives an algorithm for enumerating all the bounded cells in $\mathcal{A}_{d,d+2}$. We have:

Proposition 5.1.3 *There are exactly 2 simplex cells in $\mathcal{A}_{d,d+2}$ for $d \geqslant 2$.*

Proof. We prove it by induction on d. As figure 5.4 shows, there are 2 simplex cells in $\mathcal{A}_{2,4}$. Suppose that there are exactly 2 simplices in $\mathcal{A}_{d',d'+2}$ for $d' < d$. As the proof of proposition 5.1.2 shows, $\mathcal{A}_{d,d+2}$ has one simplex S and all the other bounded cells are lifted from an $\mathcal{A}_{d-1,d+1}$ base about an $\mathcal{A}_{d-2,d}$ axis. Each bounded cell of the axis is a facet of a distinct bounded cell of the base. Since $\mathcal{A}_{d,d+1}$ has one more facet than $\mathcal{A}_{d-2,d}$, one bounded cell in the base is not touching the axis. The bounded cell has to be a simplex, otherwise it will be lifted to be a non-simplex prism with more than $d+2$ facets, which is impossible. By the induction hypothesis, the base has exactly 2 simplices, one of which is lifted to be a simplex prism as stated above. The other simplex cell of the base has a facet on the axis and is hence lifted to be a simplex cell S' in $\mathcal{A}_{d,d+2}$. Obviously any non-simplex cell in the base will not be lifted to be simplex cell in $\mathcal{A}_{d,d+2}$. Therefore, S and S' are the only 2 simplex cells in $\mathcal{A}_{d,d+2}$. □

There is a clear one-one correspondence between the simplex cells and the simplex prism cells in $\mathcal{A}_{d,d+2}$ for $d \geq 3$. Thus, we have:

Lemma 5.1.4 *There are exactly 2 simplex prism cells in $\mathcal{A}_{d,d+2}$ for $d \geq 3$.*

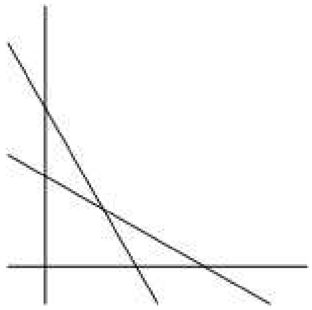

Figure 5.4: $\mathcal{A}_{2,4}$

Remark 5.1.5 *There is only 1 simplex prism cell in $\mathcal{A}_{2,4}$. Note that in dimension 2, a simplex prism is also a square and it is associated to both simplex cells.*

Let $\mathcal{A}_{d,d+2}$ be a simple arrangement formed by $d+2$ hyperplanes in dimension d. Besides simplices, the bounded cells of $\mathcal{A}_{d,d+2}$ are simple polytopes with $d+2$ facets. The $\lfloor \frac{d}{2} \rfloor$ combinatorial types of simple polytopes with $d+2$ facets are well-known, see for example [21]. We briefly recall the combinatorial structure of $\mathcal{A}_{d,d+2}$ as some of the notions presented are used in Section 5.2. As there is only one combinatorial type of simple arrangement with $d+2$ hyperplanes, the arrangement $\mathcal{A}_{d,d+2}$ can be obtained from the simplex $\mathcal{A}_{d+1,d}$ by cutting off one its vertices v with the hyperplane h_{d+2}. A prism P with a simplex base is created. Let us call *top base* the base of P which belongs to h_{d+2} and assume, without loss of generality, that the hyperplane containing the bottom base of P is h_{d+1}. Besides the simplex defined by v and the vertices of the top base of P, the remaining d bounded cells of $\mathcal{A}_{d,d+2}$ are between h_{d+2} and h_{d+1}. See Figure 5.2 for an illustration the combinatorial structure of $\mathcal{A}_{3,5}$. As the projection of $\mathcal{A}_{d,d+2}$ on h_{d+1} is combinatorially equivalent to $\mathcal{A}_{d-1,d+1}$, the d bounded cells between h_{d+2} and h_{d+1} can be obtained from the d bounded cells of $\mathcal{A}_{d-1,d+1}$ by the *shell-lifting* of $\mathcal{A}_{d-1,d+1}$ over the ridge $h_{d+1} \cap h_{d+2}$; that is, besides the vertices belonging to $h_{d+1} \cap h_{d+2}$, all the vertices in h_{d+1} (forming $\mathcal{A}_{d-1,d+1}$) are lifted. See Figure 5.5 where the skeletons of the $d+1$ bounded cells of $\mathcal{A}_{d,d+2}$ are given for $d = 2, 3, \ldots, 6$. The shell-lifting of the bounded cells is indicated by an arrow, the vertices not belonging to h_{d+1} are represented in black and the simplex cell containing v is the one made of black vertices. The bounded cells of $\mathcal{A}_{d,d+2}$ are 2 simplices and a pair of each of the $\lfloor \frac{d}{2} \rfloor$ combinatorial types of simple polytopes with $d+2$ facets for odd d. For even d one of the combinatorial type is present only once. Since all the simple polytopes with $d+2$ facets have diameter 2, we have $\delta(\mathcal{A}_{d,d+2}) = \frac{2+2(d-1)}{d+1}$.

Proposition 5.1.6 *As there is only one combinatorial type of simple arrangement with $d+2$ hyperplanes, we have $\Delta_{\mathcal{A}}(d, d+2) = \delta(\mathcal{A}_{d,d+2}) = \frac{2d}{d+1}$.*

5.2 Hyperplane arrangements with large average diameter

The arrangements $\mathcal{A}^*_{2,n}$ and $\mathcal{A}^*_{3,n}$ presented in Sections 3 and 4 can be generalized to the arrangement $\mathcal{A}^*_{d,n}$ formed by the following n hyperplanes h^d_k for $k = 1, 2, \ldots, n$. The hyperplanes $h^d_k = \{x : x_{d+1-k} = 0\}$ for $k = 1, 2, \ldots, d$ form the positive orthant, and the hyperplanes h^d_k for $k = d+1, \ldots, n$ are defined by their intersections with the axes $\bar{x}_i$ of the positive orthant. We have $h^d_k \cap \bar{x}_i = \{0, \ldots, 0, 1 + (d-i)(k-d-1)\varepsilon, 0, \ldots, 0\}$ for $i = 1, 2 \ldots, d-1$ and $h^d_k \cap \bar{x}_d = \{0, \ldots, 0, 1 - (k-d-1)\varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < \frac{1}{n-d-1}$. The combinatorial structure of $\mathcal{A}^*_{d,n}$ can be derived inductively.

All the bounded cells of $\mathcal{A}^*_{d,n}$ are on the positive side of h^d_1 and h^d_2 with the bounded cells between h^d_2 and h^d_3 being obtained by the shell-lifting of a combinatorial equivalent of $\mathcal{A}^*_{d-1,n-1}$ over the ridge $h^d_2 \cap h^d_3$, and the bounded cells on the other side of h^d_3 forming a combinatorial equivalent of $\mathcal{A}^*_{d,n-1}$. The intersection $\mathcal{A}^*_{d,n} \cap h^d_k$ is combinatorially equivalent to $\mathcal{A}^*_{d-1,n-1}$ for $k = 2, 3, \ldots, d$ and removing h^d_2 from $\mathcal{A}^*_{d,n}$ yields an arrangement combinatorially equivalent to $\mathcal{A}^*_{d,n-1}$. See Figure 4.3 for an arrangement combinatorially equivalent to $\mathcal{A}^*_{3,6}$.

Proposition 5.2.1 *The arrangement $\mathcal{A}^*_{d,n}$ contains $\binom{n-d}{d}$ cubical cells for $n \geq 2d$.*

Proof. The arrangements $\mathcal{A}^*_{2,n}$ and $\mathcal{A}^*_{3,n}$ contain, respectively, $\binom{n-d}{2}$ and $\binom{n-d}{3}$ cubical cells. The arrangement $\mathcal{A}^*_{d,2d}$ has one cubical cell. As $\mathcal{A}^*_{d,n}$ is obtained inductively from $\mathcal{A}^*_{d,n-1}$ by raising $\mathcal{A}^*_{d-1,n-1}$ over the ridge $h^d_2 \cap h^d_3$, we count separately the cubical cells between h^d_2 and h^d_3 and the ones on the other side of h^d_3. The ridge $h^d_2 \cap h^d_3$ is an hyperplane of the arrangements $\mathcal{A}^*_{d,n} \cap h^d_2$ and $\mathcal{A}^*_{d,n} \cap h^d_3$ which are both combinatorially equivalent to $\mathcal{A}^*_{d-1,n-1}$. Removing h^{d-1}_2 from $\mathcal{A}^*_{d,n} \cap h^d_2$ yields an arrangement combinatorially equivalent to $\mathcal{A}^*_{d-1,n-2}$. It implies that $\binom{(n-2)-(d-1)}{d-1}$ cubical cells of $\mathcal{A}^*_{d,n} \cap h^d_2$ are not incident to the ridge $h^d_2 \cap h^d_3$. The shell-lifting of these $\binom{n-d-1}{d-1}$ cubical cells (of dimension $d-1$) creates $\binom{n-d-1}{d-1}$ cubical cells between h^d_2 and h^d_3. As removing h^d_2 from $\mathcal{A}^*_{d,n}$ yields an arrangement combinatorial equivalent to $\mathcal{A}^*_{d,n-1}$, there are $\binom{n-1-d}{d}$ cubical cells on the other side of h^d_3. Thus, $\mathcal{A}^*_{d,n}$ contains $\binom{n-d-1}{d-1} + \binom{n-d-1}{d} = \binom{n-d}{d}$ cubical cells. $\square$

Remark 5.2.2 *A simple hyperplane arrangement $\mathcal{A}^*_{d,n}$ is combinatorially equivalent to the cyclic hyperplane arrangement, see [20] for some combinatorial properties of the (projective) cyclic hyperplane arrangement. The combinatorics of the addition of a (pseudo) hyperplane to the cyclic hyperplane arrangement is studied in details in [37]. For example, the arrangements $\mathcal{A}^*_{2,6}$ and $\mathcal{A}^o_{2,6}$ correspond to the top and bottom elements of the higher Bruhat order $B(5,2)$ given in Figure 3 of [37].*

Corollary 5.2.3 *We have $\delta(\mathcal{A}^*_{d,n}) \geq \frac{d\binom{n-d}{d}}{\binom{n-1}{d}}$ for $n \geq 2d$. It implies that for d fixed, $\Delta_{\mathcal{A}}(d,n)$ is arbitrarily close to d for n large enough.*

Similarly we can inductively count $(n-d)$ simplices and $(n-d)(n-d-1)$ bounded cells of $\mathcal{A}^*_{d,n}$ combinatorially equivalent to a prism with a simplex base. We have $(n-1)-(d-1)$ simplices in $\mathcal{A}^*_{d,n} \cap h^d_2$ and, since removing h^{d-1}_2 from $\mathcal{A}^*_{d,n} \cap h^d_2$ yields an arrangement combinatorially equivalent to $\mathcal{A}^*_{d-1,n-2}$, only one of these $(n-d)$ simplices of $\mathcal{A}^*_{d,n} \cap h^d_2$ is incident to the ridge $h^d_2 \cap h^d_3$. Thus, between h^d_2 and h^d_3, we have one simplex incident to the ridge $h^d_2 \cap h^d_3$ and $(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base not incident to the ridge $h^d_2 \cap h^d_3$. In addition, $(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base are incident to the ridge $h^d_2 \cap h^d_3$ and between h^d_2 and h^d_3. These $(n-d-1)$ cells correspond to the truncations of the simplex $\mathcal{A}^*_{d,d+1}$ by h^d_k for $k = d+2, d+3, \ldots, n$. Thus, we have $2(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base between h^d_2 and h^d_3. As the other side of h^d_3 is combinatorially equivalent to $\mathcal{A}^*_{d,n-1}$, it contains $(n-1-d)$ simplices and $(n-d-1)(n-d-2)$ bounded cells combinatorially equivalent to a prism with a simplex base. Thus, $\mathcal{A}^*_{d,n}$ has $(n-d-1)(n-d-2)+2(n-d-1) = (n-d)(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base and $(n-d)$ simplices. As a prim with a simplex base has diameter 2 and the diameter of a bounded cell is at least 1, Corollary 5.2.3 can be slightly strengthened to the following corollary.

Corollary 5.2.4 *We have* $\Delta_{\mathcal{A}}(d,n) \geq 1 + \frac{(d-1)\binom{n-d}{d}+(n-d)(n-d-1)}{\binom{n-1}{d}}$ *for* $n \geq 2d$.

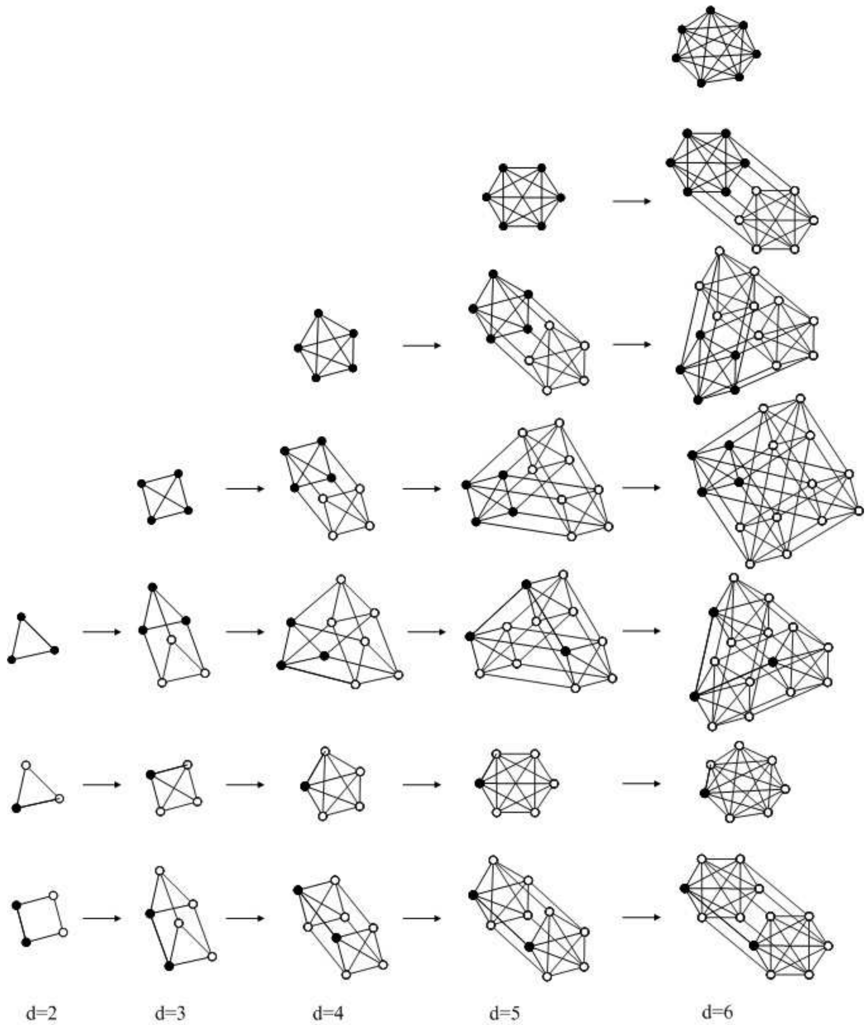

Figure 5.5: The skeletons of the $d+1$ bounded cells of $\mathcal{A}_{d,d+2}$ for $d = 2, 3, \ldots, 6$.

Chapter 6

Computational Approach

Along the theoretic approach, we developed code to check the conjectures and provide some insight into the combinatorial structure of the hyperplane arrangement.

Computer plays an important role in enumerative combinatorics, as the size of a problem usually grows too fast to be handled by human. The enumeration of the combinatorial types of simple arrangement ($\mathcal{A}_{d,n}$) is very expensive even for relatively small n and d. Another difficulty arises from the numerical problems in floating point computations.

6.1 Enumeration of simple hyperplane arrangements

To enumerate arrangements directly is considered to be very hard. To the best of our knowledge, it has to done indirectly; that is, by first enumerating a combinatorial abstract generalization, i.e. oriented matroids, and then returning to hyperplane arrangements, see [18].

Simple arrangements with 6 and 7 lines ($\mathcal{A}_{2,6}$ and $\mathcal{A}_{2,7}$), with 6 planes

($\mathcal{A}_{3,6}$) are enumerated[1] and analyzed. These computations helped us to identify and understand some of their combinatorial properties. The enumeration of the oriented matroids is based on Finschi's online database [30], presented as chirotopes (RevLex-Index) . For the theoretical background and computational framework for generating oriented matroids, see [19, 17]. The Perl codes written by Nakayama [28] are used to convert the chirotopes to hyperplane arrangement representations, as a list of the equations of the hyperplanes forming the arrangement [27]. See Appendix B.2 for an illustration.

6.2 Enumeration of the bounded cells of hyperplane arrangements

Given a simple arrangement $\mathcal{A}_{d,n}$, we are mainly interested in its average diameter, its combinatorial type, and its bounded cells. Therefore, one key computation is the enumeration of the bounded cells of a hyperplane arrangement.

Edelsbrunner describes in Chapter 7 of [14] an incremental algorithm of arrangement construction, which lists all the faces (including cells) as well as their incidences. The running time is $O(n^d)$, which is optimal as $\mathcal{A}_{d,n}$ has $\Omega(n^d)$ faces.

There also exists a reverse search algorithm [2] that is dedicated to arrangement cell enumeration [36]. The running time is $O(n \cdot |C|)$, where C is the set of cells. According to Theorem 1.2.4, in $\mathcal{A}_{d,n}$, $|C| = \sum_{i=0}^{d} \binom{n}{i} = \Theta(n^d)$. So, in our case the cost is $O(n^{d+1})$.

In terms of efficiency, the above algorithms are both good candidates for the cells enumeration needed for this thesis. However, efficiency is not our

[1] Our enumeration of $\mathcal{A}_{2,7}$ is not yet complete.

main consideration as we currently consider low dimensions and small number of hyperplanes. Weibel's *minksum* package [25, 16], which is very handy, gives the sign vectors associated with the cells, making it easier to analyze the cells combinatorial properties. In order to use *minksum*, which is originally meant for polyhedral computation, the relationship between linear arrangements and zonotopes described in Section 1.4 is exploited. Followed are some details about *minksum*.

Given a d-dimensional arrangement $\mathcal{A}_{d,n}$ which is formed by hyperplanes $h_1, h_2, \cdots, h_n$, it is first transformed to a linear arrangement by *dimension lifting* $\mathcal{A}_{d,n}$ to $\mathcal{A}_{d+1,n}$ formed by $h'_1, h'_2, \cdots, h'_n$ (see Figure 5.3): $\mathcal{A}_{d,n}$ is lifted to plane $x_{d+1} = 1$ in $I\!R^{d+1}$ and the hyperplane h'_i in $\mathcal{A}_{d+1,n}$ contains the origin and h_i in $\mathcal{A}_{d,n}$ $(i = 1, 2, \cdots, n)$. See Appendix B.1 for the computational aspects of dimension lifting.

After the dimension lifting procedure, the normal vectors of $\mathcal{A}_{d+1,n}$, which can also be viewed as line segments, are given as input to *minksum* to compute the Minkowski sum of the line segments, which is a zonotope. By Proposition 1.4.2, this zonotope corresponds in the dual space to $\mathcal{A}_{d+1,n}$. So *minksum* outputs the combinatorial structure of $\mathcal{A}_{d+1,n}$ including the sign vectors for each cell of $\mathcal{A}_{d+1,n}$. Since $\mathcal{A}_{d+1,n}$ is linear, each cell is a cone that is pointed at the origin. To get the original cell in $\mathcal{A}_{d,n}$, we use sign vectors to find the bounding hyperplanes and use *cdd* [9] library to get the V-representation of the cell, as well as the adjacency information between vertices and facets. See Appendix B.3 for an illustration of minksum computation.

Remark 6.2.1 *While we use minksum just for line segments (1-dimensional polytopes), minksum can compute Minkowski sum of polytopes in general di-*

mensions.

6.3 Average diameter computation

The pseudo-code for the computation of average diameter δ of a given arrangement $\mathcal{A}_{d,n}$ is given in Algorithm 1. $\mathcal{A}_{d,n}$ is represented as a list of n hyperplane equations. Besides δ, the set of bounded cells F_d^+ (in terms of sign vectors) and the number of external facets $f_{d-1}^0(\mathcal{A}_{d,n})$ are also computed.

First $\mathcal{A}_{d,n}$ is dimension-lifted to a linear arrangement $\mathcal{A}_{d+1,n}$ (see Appendix B.1). Then, as presented in Section 6.2, *minksum* package is used to enumerate the cells of $\mathcal{A}_{d+1,n}$ in the form of sign vectors. Based on each sign vector, we can get the inequalities (H-representation) that define the corresponding cell. Once we obtained the list of the bounded cells as H-representations, we simply consider their skeletons given as byproducts of the conversion from H-representation to V-representation. Finally, we use classic graphs algorithms[2] to compute the average diameter.

[2]For graph algorithms, the Python library NetworkX is used.

Algorithm 1: AverageDiameter

input : $\mathcal{A}_{d,n}$
output: $\delta, F_d^+, f_{d-1}^0(\mathcal{A}_{d,n})$

Initialize $\delta, F_d^+, f_{d-1}^0(\mathcal{A}_{d,n})$;
$\mathcal{A}_{d+1,n} \leftarrow$ `DimensionLift` $(\mathcal{A}_{d,n})$;
SignVectors $\leftarrow$ `EnumerateCells` $(\mathcal{A}_{d+1,n})$; `/* minksum */`
foreach SignVector $\in$ SignVectors **do**
  if `IsBounded` $(\mathcal{A}_{d,n}$, SignVector$)$ **then**
    G $\leftarrow$ `GetSkeleton` $(\mathcal{A}_{d,n}$, SignVector$)$;
    `UpdateAverageDiameter` $(\delta$, `Diameter` (G));
    $F_d^+ \leftarrow F_d^+ \cup$ SignVector;
    `/* get the indices of the hyperplanes that tightly bound the cell: */`
    IND $\leftarrow$ `RemoveRedundancy` $(\mathcal{A}_{d,n}$, SignVector$)$;
    foreach ind $\in$ IND **do**
      `/* invert the ind'th sign of` SignVector `to get the neighboring cell: */`
      SignVectorNeighbor $\leftarrow$ `Invert` (SignVector, ind);
      if `IsBounded` $(\mathcal{A}_{d,n}$, SignVectorNeighbor$)$ **then**
        $f_{d-1}^0(\mathcal{A}_{d,n}) \leftarrow f_{d-1}^0(\mathcal{A}_{d,n}) + 1$;
return $\delta, F_d^+, f_{d-1}^0(\mathcal{A}_{d,n})$;

For a cell P given as H-representation, the *cdd* package is used to check whether P is bounded or not. In the computation for the V-representation of P, *cdd* calculates both the extreme points $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_p}\}$ and the rays $\{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_q}\}$ that defines P, i.e.,

$$P = conv(\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_p}) + cone(\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_q}),$$

where *conv* stands for *the convect hull of*, and *cone* for *the conical hull of*. The unboundedness of P is indicated by t he presence of rays in the V-representation, see Algorithm 2.

Algorithm 2: IsBounded

input : $\mathcal{A}_{d,n}$, SignVector.
output: True if the cell represented by SignVector is bounded, otherwise False.

```
Hrep ← GetHRepresentation (A_{d,n}, SignVector);
Vrep ← GetVRepresentation (Hrep);
if HasRay (Vrep) then return False else return True;
```

Redundancy removal is also one of the options of the *cdd* package[3]. Each facet of the cell P corresponds to an inequality that tightly bounds P. Let the inequality related to some facet F be the ith one. Then inverting the ith sign of P's sign vector yields the neighboring cell P' that shares the facet F with P. Recall that a facet is external if it belongs to exactly one bounded cell. So, whether F is external can be determined by checking the boundedness of P and P'.

The algorithm is implemented using Python [31], a scripting language influenced by Perl. As a "glue language", its strong text processing capability makes it ideal for this project, in which most of the computation extensive tasks are taken over by existing softwares (CDD, minksum, Nakayama's codes) with different input/output formats. Additionally, Python's loose syntax and rich external modules (e.g. NetworkX for graph handling) are quite convenient. My choice of Python, among similar languages, was motivated by my previous experience with Python.

[3]Redundancy removal is only available in *cdd+*.

Chapter 7

Concluding Remarks

We investigate combinatorial properties of hyperplane arrangements with a focus on estimating the average diameter of a bounded cell and the complexity of the envelope of simple arrangements defined by n hyperplanes in dimension d. In particular we substantiate two recent conjectures of Deza, Terlaky and Zinchenko.

We provide exact values for the average diameter in the plane, and asymptotically equal upper and lower bounds in dimension 3. In general, we provide a lower bound arbitrarily close to the dimension d as n goes to infinity. If Hirsch conjecture holds, the upper and lower bounds are asymptotically equal in fixed dimension.

Besides Hirsch conjecture, we discuss links with a recent result of Dedieu, Malajovich and Shub, and with a result of Haimovich. Computational tools to generate and compute hyperplane arrangements are presented together with preliminary computational results.

Future works include further efforts to exploit Haimovich's result and oriented matroids' rich theory. One natural generalization is to look at the same conjectures of Deza, Terlaky and Zinchenko for oriented matroids instead of hyperplane arrangements. It would be also important to tackle numerical problems which occur as the dimension and number of hyperplanes increase. Adopting rational number computation would be our first step.

Appendix A

Theoretical Framework

A.1 Example oriented matroids

We illustrate some of properties of oriented matroids by few examples. For an example of orient matroid associated to vector configuration, see Example 1. A *point configuration* $X = \{\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}\} \subseteq I\!R^d$ is a set of finitely many points in affine space $I\!R^d$. By abuse of notation, X also denotes the $r \times n$ matrix $[\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}]$. An *affine dependency* of X is a vector $\mathbf{z} \in I\!R^n$ satisfying $\mathbf{1}^T\mathbf{z} = 0$ and $X\mathbf{z} = \mathbf{0}$.

Example 3 (oriented matroid associated with point configuration) *Let $\mathcal{M} = (E, \mathcal{C})$ be the oriented matroid associated with the point configuration $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$. Then E is the set of points. The set of circuits $\mathcal{C}$ includes the sign vectors of the minimal affine dependencies. For example, $\begin{pmatrix} \frac{1}{3} \\ -1 \\ \frac{2}{3} \\ 0 \end{pmatrix}$ is a minimal affine dependency and its sign vector $\begin{pmatrix} + \\ - \\ + \\ 0 \end{pmatrix}$ is one of the circuits of $\mathcal{M}$.*

$\mathcal{M}$ is the same as the oriented matroid of the vector configuration in Ex-

ample 1 as the point configuration can be obtained from the vector configuration by the mapping $\mathbf{v} \to \frac{\mathbf{v}}{\mathbf{c}\cdot\mathbf{v}}$, *where* $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *(see Figure A.1).*

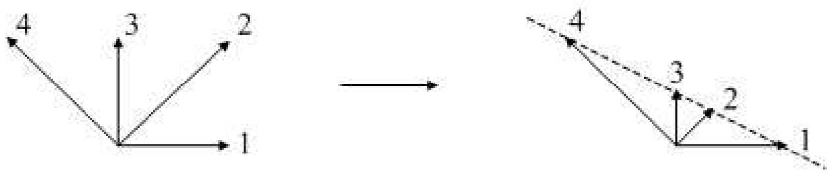

Figure A.1: The vector configuration (left) and the point configuration (right) have the same oriented matroid.

The oriented matroid associated to a linear hyperplane arrangement corresponds naturally to the oriented matroid of the vector configuration consisting of the normal vectors of the hyperplanes.

A.2 Permutahedron of order 4 - an example of zonotope

A permutahedron of order 4 is the convex hull of the permutations of $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and form a 3-polytope, see Figure A.2[1].

Being a zonotope, the permutahedron is

(1) the image of 6-cube under affine projection $\pi : I\!R^6 \to I\!R^4, \pi(\mathbf{x}) = A\mathbf{x} + \frac{5}{2}\mathbf{1}$,

[1]picture courtesy of http://www.antiquark.com.

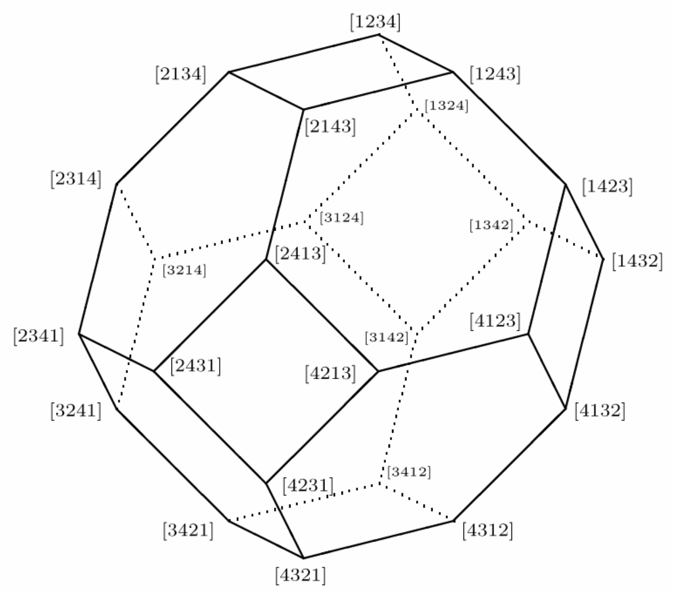

Figure A.2: Permutahedron of order 4.

where

$$A = \left[\frac{\mathbf{e_2}-\mathbf{e_1}}{2}, \frac{\mathbf{e_3}-\mathbf{e_1}}{2}, \frac{\mathbf{e_4}-\mathbf{e_1}}{2}, \frac{\mathbf{e_3}-\mathbf{e_2}}{2}, \frac{\mathbf{e_4}-\mathbf{e_2}}{2}, \frac{\mathbf{e_4}-\mathbf{e_3}}{2}\right]$$
$$= \frac{1}{2}\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

, and

(2) the Minkowski sum of the following 6 line segments:

$$\left[-\frac{\mathbf{e_2}-\mathbf{e_1}}{2}, \frac{\mathbf{e_2}-\mathbf{e_1}}{2}\right], \left[-\frac{\mathbf{e_3}-\mathbf{e_1}}{2}, \frac{\mathbf{e_3}-\mathbf{e_1}}{2}\right], \left[-\frac{\mathbf{e_4}-\mathbf{e_1}}{2}, \frac{\mathbf{e_4}-\mathbf{e_1}}{2}\right],$$
$$\left[-\frac{\mathbf{e_3}-\mathbf{e_2}}{2}, \frac{\mathbf{e_3}-\mathbf{e_2}}{2}\right], \left[-\frac{\mathbf{e_4}-\mathbf{e_2}}{2}, \frac{\mathbf{e_4}-\mathbf{e_2}}{2}\right], \left[-\frac{\mathbf{e_4}-\mathbf{e_3}}{2}, \frac{\mathbf{e_4}-\mathbf{e_3}}{2}\right].$$

The permutahedron is associated, through normal fan, to the arrangement formed by 6 linear hyperplanes whose norm vectors are

$$\frac{\mathbf{e_2} - \mathbf{e_1}}{2}, \frac{\mathbf{e_3} - \mathbf{e_1}}{2}, \frac{\mathbf{e_4} - \mathbf{e_1}}{2}, \frac{\mathbf{e_3} - \mathbf{e_2}}{2}, \frac{\mathbf{e_4} - \mathbf{e_2}}{2}, \frac{\mathbf{e_4} - \mathbf{e_3}}{2}.$$

Appendix B

Computational Framework

B.1 Hyperplane dimension lifting

Given a d-dimensional hyperplane $a_1x_1 + a_2x_2 + \cdots + a_dx_d = b$, it can be lifted along the extra dimension such that the $(n+1)$th coordinate x_{d+1} of every point in the hyperplane is the same. For simplicity, let $x_{d+1} = 1$. The norm of the $(d+1)$-dimensional hyperplane that contains the origin and the lifted d-dimensional hyperplane is $(a_1, a_2, \cdots, a_n, -b)$. We illustrate this property for the dimension lifting from dimension 2 to 3. Let $a_1x_1 + a_2x_2 = b$ be the equation of the line to be lifted. Then the intersection of the line with x_1 and x_2 axes are $\frac{b}{a_1}$ and $\frac{b}{a_2}$ respectively. According to Figure B.1, we have

$$
\begin{aligned}
\vec{AB} &= \left(\tfrac{b}{a_1}, -\tfrac{b}{a_2}\right)\\
\vec{A'B'} &= \left(\tfrac{b}{a_1}, -\tfrac{b}{a_2}, 0\right)\\
\vec{OA'} &= \left(\tfrac{b}{a_1}, 0, 1\right).
\end{aligned}
$$

The normal vector of the plane $OA'B'$ is

$$
\begin{aligned}
& \vec{OA'} \times \vec{A'B'}\\
= & \left(\tfrac{b}{a_1}, 0, 1\right) \times \left(\tfrac{b}{a_1}, -\tfrac{b}{a_2}, 0\right)\\
= & \left(\tfrac{b}{a_2}, \tfrac{b}{a_1}, -\tfrac{b^2}{a_1a_2}\right),
\end{aligned}
$$

which can be formulated to $(a_1, a_2, -b)$.

The formula for general dimensions can be obtained similarly.

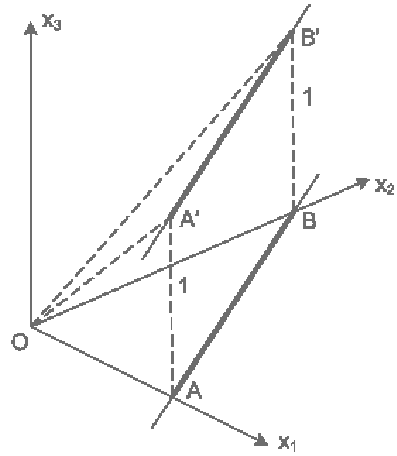

Figure B.1: Hyperplane dimension lifting from dimension 2 to 3.

B.2 Oriented matroid and arrangement representation

Given an oriented matroid as a chirotope, Hiroki Nakayama's codes are used to check its realizability and, when realizable, to convert it to the corresponding arrangement representation. The codes are based on Nakayama's PhD dissertation [27]. We illustrate this procedure by showing how the representation of $\mathcal{A}^o_{2,6}$ is obtained.

All chirotopes corresponding to $\mathcal{A}_{2,6}$ can be found in the catalog of hyperplane arrangements (rank=3, card=7) in Finschi's online database [30]. The chirotope of $\mathcal{A}^o_{2,6}$ is

```
11121121231121231234112123123412345
22332334442334445555233444555566666
34445555556666666666777777777777777

++++++++++++++++++++++++++++++-++++.
```

There are $\binom{7}{3} = 35$ signs, which are in reverse lexicographic order of their indices. The only negative sign is the one that is associated with the determinant of the matrix formed by vectors 1, 6 and 7, denoted by $[167] = -$.

Recall that Realizing the oriented matroid is equivalent to finding vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_7} \in I\!R^3$ such that the signs of the maximal minors of the matrix $[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_7}]$ confirm with the chirotope. Since at least one of the signs is $+$, we can assume that one of the 3×3 submatrix is the identity matrix, i.e., 3 of the vectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The minimal reduced system of the chirotope is computed first. It consists of $[237], [234], [346], [347], [267], [167], [567], [467], [157], [123], [367], [456], [345]$. Then the realizability of the oriented matroid is checked by computing its solvability sequence. If realizable, each unknown variable in the above matrix is assigned with a proper value and the vector configuration is obtained:

$$\begin{bmatrix} -1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & \frac{1}{2} & 1 & 1 & 0 \\ 6 & \frac{5}{2} & 0 & -\frac{3}{4} & -1 & 0 & 1 \end{bmatrix}. \tag{2.2.1}$$

The last step is to convert the vector configuration in dimension 3 to the corresponding arrangement in dimension 2, which is straightforward as the configuration is naturally associated with a linear hyperplane arrangement by identifying the vectors as the normal vectors of the hyperplanes in the arrangement. The 3-dimensional linear hyperplane arrangement is then converted to an plane arrangement by a a reversed procedure of dimension lifting described in Appendix B.1. The equations of the 6 lines (one of the vectors corresponding to the infinity hyperplane) in $\mathcal{A}^o_{2,6}$ are

$$\begin{array}{rcl} -x-2y+6 & = & 0 \\ x-y+\frac{5}{2} & = & 0 \\ x & = & 0 \\ x+\frac{1}{2}y-\frac{3}{4} & = & 0 \\ x+y-1 & = & 0 \\ y & = & 0. \end{array}$$

B.3 Minksum package

minksum [25, 16] is a software package that computes the Minkowski sum of polytopes. We illustrate how we used for arrangement computation.

Let $\mathcal{A}$ be a 2D linear arrangement formed by lines $x = 0$, $y = 0$ and $x + y = 0$, whose normal vectors are $(0, 1)$, $(1, 0)$ and $(1, 1)$ respectively. In order to get the sign vectors of each cell in $\mathcal{A}$, 3 line segments indicated by the normal vectors are used as the input to *minksum*. Followed are the the input and output.

```
INPUT:
3
[[0,0],[1,0]]
[[0,0],[0,1]]
[[0,0],[1,1]]

OUTPUT:
[1,2,1] : [0,1] : [-2,1]
```

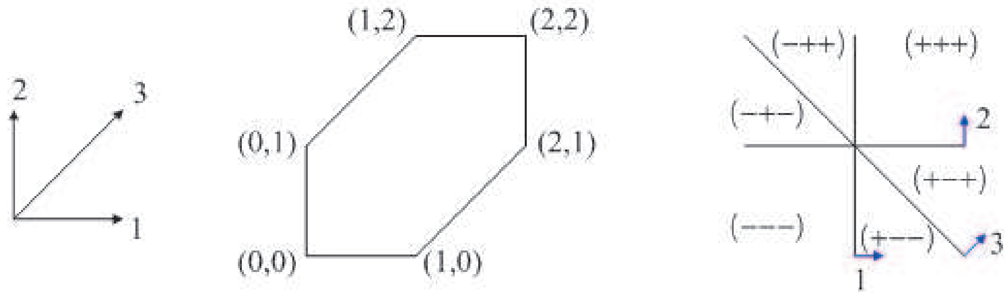

Figure B.2: Example of using *minksum*.

```
[1,2,2] : [1,2] : [-1,2]
[1,1,1] : [0,0] : [-1,-1]
[2,1,1] : [1,0] : [1,-2]
[2,1,2] : [2,1] : [2,-1]
[2,2,2] : [2,2] : [1,1]
```

The first column of the output lists the sign vectors of the 6 cells of $\mathcal{A}$ ("1" for "-", "2" for "+"); the third column gives a point in each cell. The Minkowski sum is a zonotope with 6 vertices, which are listed in the second column.

Figure B.2 illustrates the Minkowski sum and what is happening in the dual space.

Appendix C

Preliminary Computational Results

In the following tables, the first column are the ID's of the simple arrangement, which are actually the sequence numbers of the arrangements in the enumeration. For each arrangement, we list its diameter, the number of each type of its bounded cells and the number of its external facets.

C.1 Line arrangements with 6 lines

We use $f^+_{2,i}(\mathcal{A}_{2,6})$ $(3 \leq i \leq 6)$ to denote the number of bounded cells with i facets, or polygons with i edges. Notice that $\sum_{i=3}^{6} f^+_{2,i}(\mathcal{A}_{2,6}) = f^+_2(\mathcal{A}_{2,6}) = 10$.

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 6)$	f^0_1
1	1.5	5,3,2,0	11
2	1.6	4,5,1,0	11
3	1.5	5,3,2,0	12
4	1.6	4,6,0,0	12
5	1.6	4,5,1,0	11
6	1.5	5,4,1,0	12
7	1.5	5,3,2,0	11
8	1.6	4,6,0,0	12
9	1.4	6,2,2,0	12
10	1.5	5,4,1,0	12
11	1.6	4,6,0,0	12
12	1.6	4,5,1,0	11
13	1.6	4,5,1,0	11
14	1.5	5,4,1,0	12
15	1.5	6,3,0,1	12
16	1.3	7,0,3,0	12
17	1.4	6,3,1,0	13
18	1.4	6,3,1,0	13
19	1.5	6,3,0,1	12
20	1.5	5,3,2,0	11
21	1.5	5,4,1,0	12
22	1.6	4,5,1,0	11

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 6)$	f^0_1
23	1.6	4,6,0,0	12
24	1.4	6,2,2,0	12
25	1.5	5,3,2,0	11
26	1.5	5,4,1,0	12
27	1.6	4,6,0,0	12
28	1.6	4,4,2,0	10
29	1.6	4,6,0,0	12
30	1.6	4,5,1,0	11
31	1.6	4,6,0,0	12
32	1.5	5,3,2,0	11
33	1.6	4,6,0,0	12
34	1.5	5,4,1,0	12
35	1.5	5,4,1,0	12
36	1.6	4,6,0,0	12
37	1.6	5,4,0,1	11
38	1.6	4,6,0,0	12
39	1.6	4,5,1,0	11
40	1.6	4,6,0,0	12
41[1]	**1.7**	**4,5,0,1**	**10**
42	1.6	4,6,0,0	12
43[2]	1.6	4,6,0,0	12

C.2 Line arrangements with 7 lines

We use $f^+_{2,i}(\mathcal{A}_{2,7})$ $(3 \leq i \leq 7)$ to denote the number of bounded cells with i facets, or polygons with i edges. Note that $\sum_{i=3}^{7} f^+_{2,i}(\mathcal{A}_{2,7}) = f^+_2(\mathcal{A}_{2,7}) = 15$.

[1] $\mathcal{A}^o_{6,2}$
[2] $\mathcal{A}^*_{6,2}$

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
1	1.60	6,7,2,0,0	14
2	1.60	6,6,3,0,0	13
3	1.60	6,7,2,0,0	14
4			
5	1.67	5,8,2,0,0	13
6	1.67	5,9,1,0,0	14
7	1.67	5,9,1,0,0	14
8	1.60	6,7,2,0,0	14
9	1.60	6,7,2,0,0	14
10	1.60	6,7,2,0,0	14
11			
12			
13	1.53	7,5,3,0,0	14
14	1.53	7,5,3,0,0	14
15	1.67	5,9,1,0,0	14
16	1.60	6,7,2,0,0	14
17	1.53	7,6,2,0,0	15
18	1.60	6,7,2,0,0	14
19	1.53	7,5,3,0,0	14
20			
21	1.47	8,3,4,0,0	14
22	1.40	9,3,3,0,0	16
23			
24	1.40	10,1,3,1,0	15
25			
26	1.47	8,5,2,0,0	16
27	1.53	7,6,2,0,0	15
28	1.47	8,5,2,0,0	16
29			
30			
31	1.47	9,2,3,1,0	14
32	1.47	9,3,2,1,0	15
33	1.47	8,4,3,0,0	15
34	1.47	8,4,3,0,0	15
35	1.47	8,4,3,0,0	15

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
36	1.53	7,5,3,0,0	14
37	1.47	8,5,2,0,0	16
38	1.53	8,3,3,1,0	13
39	1.47	9,3,2,1,0	15
40	1.47	8,4,3,0,0	15
41	1.47	8,4,3,0,0	15
42	1.47	8,5,2,0,0	16
43	1.47	8,3,4,0,0	14
44			
45	1.47	8,3,4,0,0	14
46	1.47	8,4,3,0,0	15
47			
48			
49			
50	1.47	9,2,3,1,0	14
51	1.47	8,4,3,0,0	15
52			
53	1.40	10,2,2,1,0	16
54	1.40	9,4,2,0,0	17
55	1.53	7,6,2,0,0	15
56	1.47	8,4,3,0,0	15
57	1.47	9,3,2,1,0	15
58			
59			
60	1.60	6,7,2,0,0	14
61	1.53	7,5,3,0,0	14
62			
63	1.60	6,7,2,0,0	14
64	1.60	6,6,3,0,0	13
65	1.67	5,9,1,0,0	14
66	1.60	6,7,2,0,0	14
67	1.53	7,5,3,0,0	14
68	1.53	7,5,3,0,0	14
69	1.60	6,7,2,0,0	14
70	1.60	6,7,2,0,0	14

ID	δ	$f^+_{2,i}$ $(3 \le i \le 7)$	f^0_1
71			
72	1.67	5,9,1,0,0	14
73	1.67	5,9,1,0,0	14
74	1.67	5,8,2,0,0	13
75	1.67	5,8,2,0,0	13
76	1.67	5,9,1,0,0	14
77	1.67	5,9,1,0,0	14
78	1.60	6,7,2,0,0	14
79	1.60	6,6,3,0,0	13
80	1.53	7,5,3,0,0	14
81	1.60	6,8,1,0,0	15
82	1.53	7,6,2,0,0	15
83	1.60	6,6,3,0,0	13
84	1.60	6,7,2,0,0	14
85	1.60	6,7,2,0,0	14
86	1.53	7,5,3,0,0	14
87	1.47	8,4,3,0,0	15
88	1.47	8,4,3,0,0	15
89	1.47	8,3,4,0,0	14
90			
91	1.53	7,6,2,0,0	15
92	1.53	7,5,3,0,0	14
93	1.60	6,7,2,0,0	14
94			
95			
96	1.47	8,4,3,0,0	15
97			
98	1.47	8,4,3,0,0	15
99	1.47	8,3,4,0,0	14
100	1.53	7,5,3,0,0	14
101	1.53	7,6,2,0,0	15
102	1.53	8,4,2,1,0	14
103			
104			
105	1.67	5,8,2,0,0	13

ID	δ	$f^+_{2,i}$ $(3 \le i \le 7)$	f^0_1
106	1.60	6,6,3,0,0	13
107	1.60	6,8,1,0,0	15
108	1.67	5,10,0,0,0	15
109	1.60	6,7,2,0,0	14
110	1.40	9,3,3,0,0	16
111	1.47	9,3,2,1,0	15
112	1.47	9,3,2,1,0	15
113	1.40	9,3,3,0,0	16
114			
115	1.53	7,6,2,0,0	15
116	1.60	6,7,2,0,0	14
117	1.53	7,5,3,0,0	14
118	1.60	6,7,2,0,0	14
119	1.47	8,4,3,0,0	15
120	1.53	7,5,3,0,0	14
121	1.47	8,3,4,0,0	14
122	1.60	6,7,2,0,0	14
123	1.53	7,6,2,0,0	15
124	1.53	7,5,3,0,0	14
125	1.53	7,4,4,0,0	13
126	1.53	7,5,3,0,0	14
127	1.47	8,4,3,0,0	15
128	1.53	7,6,2,0,0	15
129	1.60	6,7,2,0,0	14
130	1.60	6,7,2,0,0	14
131	1.60	6,7,2,0,0	14
132	1.67	5,8,2,0,0	13
133	1.60	6,6,3,0,0	13
134	1.60	6,6,3,0,0	13
135	1.67	5,8,2,0,0	13
136	1.67	5,9,1,0,0	14
137	1.60	6,7,2,0,0	14
138	1.60	6,7,2,0,0	14
139	1.53	7,5,3,0,0	14
140	1.47	8,3,4,0,0	14

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
141	1.53	7,4,4,0,0	13
142	1.60	6,8,1,0,0	15
143	1.47	8,4,3,0,0	15
144	1.53	7,6,2,0,0	15
145	1.60	6,7,2,0,0	14
146	1.60	6,7,2,0,0	14
147	1.60	6,7,2,0,0	14
148	1.53	7,5,3,0,0	14
149	1.60	6,6,3,0,0	13
150	1.60	6,7,2,0,0	14
151	1.60	6,6,3,0,0	13
152	1.60	6,7,2,0,0	14
153	1.53	7,6,2,0,0	15
154	1.60	6,8,1,0,0	15
155	1.60	6,7,2,0,0	14
156	1.60	6,7,2,0,0	14
157	1.53	7,5,3,0,0	14
158	1.60	6,6,3,0,0	13
159	1.53	7,6,2,0,0	15
160	1.53	7,5,3,0,0	14
161	1.60	6,7,2,0,0	14
162	1.67	5,9,1,0,0	14
163	1.60	6,6,3,0,0	13
164	1.60	6,7,2,0,0	14
165	1.67	5,9,1,0,0	14
166	1.67	5,8,2,0,0	13
167	1.60	6,7,2,0,0	14
168	1.60	6,7,2,0,0	14
169	1.60	6,7,2,0,0	14
170	1.67	5,9,1,0,0	14
171	1.60	6,7,2,0,0	14
172	1.60	6,6,3,0,0	13
173	1.67	5,9,1,0,0	14
174	1.67	5,8,2,0,0	13
175	1.60	6,7,2,0,0	14

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
176	1.67	5,8,2,0,0	13
177	1.60	6,7,2,0,0	14
178	1.60	6,8,1,0,0	15
179	1.67	5,8,2,0,0	13
180	1.53	7,5,3,0,0	14
181	1.53	7,5,3,0,0	14
182	1.53	7,6,2,0,0	15
183	1.60	6,7,2,0,0	14
184	1.60	6,7,2,0,0	14
185	1.53	7,5,3,0,0	14
186	1.67	5,9,1,0,0	14
187	1.67	5,8,2,0,0	13
188	1.60	6,7,2,0,0	14
189	1.67	5,9,1,0,0	14
190	1.60	6,7,2,0,0	14
191	1.60	6,8,1,0,0	15
192	1.60	6,6,3,0,0	13
193	1.67	5,8,2,0,0	13
194	1.60	6,7,2,0,0	14
195	1.60	7,6,1,1,0	14
196	1.60	6,8,1,0,0	15
197	1.60	6,8,1,0,0	15
198	1.67	6,8,0,1,0	14
199	1.60	6,7,2,0,0	14
200	1.60	6,7,2,0,0	14
201	1.67	5,9,1,0,0	14
202	1.60	6,7,2,0,0	14
203	1.67	5,10,0,0,0	15
204	1.67	5,9,1,0,0	14
205	1.67	6,8,0,1,0	14
206	1.67	6,7,1,1,0	13
207	1.53	7,6,2,0,0	15
208	1.53	7,7,1,0,0	16
209	1.53	7,6,2,0,0	15
210	1.47	8,4,3,0,0	15

ID	δ	$f^+_{2,i}$ $(3 \le i \le 7)$	f^0_1
211	1.47	8,3,4,0,0	14
212	1.60	7,7,0,1,0	15
213	1.53	7,6,2,0,0	15
214	1.60	7,7,0,1,0	15
215	1.47	8,5,2,0,0	16
216	1.53	8,5,1,1,0	15
217	1.40	9,2,4,0,0	15
218	1.60	7,7,0,1,0	15
219	1.47	9,3,2,1,0	15
220	1.53	7,7,1,0,0	16
221	1.47	8,5,2,0,0	16
222	1.60	7,7,0,1,0	15
223	1.53	7,6,2,0,0	15
224	1.53	7,6,2,0,0	15
225	1.47	8,4,3,0,0	15
226	1.47	8,4,3,0,0	15
227	1.60	6,8,1,0,0	15
228	1.60	7,6,1,1,0	14
229	1.60	7,6,1,1,0	14
230	1.60	7,6,1,1,0	14
231	1.60	7,6,1,1,0	14
232	1.53	7,6,2,0,0	15
233	1.53	7,6,2,0,0	15
234	1.60	6,8,1,0,0	15
235	1.60	7,7,0,1,0	15
236	1.53	7,6,2,0,0	15
237	1.53	7,5,3,0,0	14
238	1.47	8,4,3,0,0	15
239	1.53	7,6,2,0,0	15
240	1.53	8,5,1,1,0	15
241	1.60	7,6,1,1,0	14
242	1.47	8,4,3,0,0	15
243	1.60	7,6,1,1,0	14
244	1.53	7,7,1,0,0	16
245	1.47	9,3,2,1,0	15

ID	δ	$f^+_{2,i}$ $(3 \le i \le 7)$	f^0_1
246	1.47	8,4,3,0,0	15
247	1.47	8,5,2,0,0	16
248	1.53	8,5,1,1,0	15
249	1.60	6,7,2,0,0	14
250	1.53	7,5,3,0,0	14
251	1.60	6,7,2,0,0	14
252	1.67	5,9,1,0,0	14
253	1.60	7,6,1,1,0	14
254	1.60	6,8,1,0,0	15
255	1.60	6,8,1,0,0	15
256	1.60	7,6,1,1,0	14
257	1.53	7,7,1,0,0	16
258	1.40	10,0,4,1,0	14
259	1.47	8,4,3,0,0	15
260	1.47	9,3,2,1,0	15
261	1.47	8,5,2,0,0	16
262	1.53	7,7,1,0,0	16
263	1.53	8,5,1,1,0	15
264	1.53	8,5,1,1,0	15
265	1.67	5,10,0,0,0	15
266	1.60	6,8,1,0,0	15
267	1.60	7,6,1,1,0	14
268	1.60	7,6,1,1,0	14
269	1.53	7,5,3,0,0	14
270	1.60	6,7,2,0,0	14
271	1.60	6,8,1,0,0	15
272	1.60	6,6,3,0,0	13
273	1.60	6,7,2,0,0	14
274	1.53	7,6,2,0,0	15
275	1.60	6,7,2,0,0	14
276	1.53	7,5,3,0,0	14
277	1.53	7,5,3,0,0	14
278	1.53	7,5,3,0,0	14
279	1.47	8,3,4,0,0	14
280	1.53	7,6,2,0,0	15

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
281	1.60	6,7,2,0,0	14
282	1.47	8,4,3,0,0	15
283			
284			
285	1.47	8,3,4,0,0	14
286			
287	1.60	6,8,1,0,0	15
288	1.53	7,6,2,0,0	15
289	1.67	5,10,0,0,0	15
290	1.60	7,5,2,1,0	13
291	1.60	6,7,2,0,0	14
292	1.60	6,7,2,0,0	14
293	1.67	5,8,2,0,0	13
294	1.60	6,6,3,0,0	13
295	1.60	6,6,3,0,0	13
296	1.67	5,9,1,0,0	14
297	1.60	6,7,2,0,0	14
298	1.67	5,10,0,0,0	15
299	1.60	6,7,2,0,0	14
300	1.47	8,4,3,0,0	15
301	1.47	8,2,5,0,0	13
302	1.47	8,4,3,0,0	15
303	1.53	7,5,3,0,0	14
304	1.47	8,3,4,0,0	14
305	1.53	7,6,2,0,0	15
306	1.53	7,6,2,0,0	15
307	1.53	7,6,2,0,0	15
308	1.60	6,7,2,0,0	14
309	1.60	6,8,1,0,0	15
310	1.60	6,8,1,0,0	15
311	1.53	7,6,2,0,0	15
312	1.60	7,5,2,1,0	13
313	1.67	6,8,0,1,0	14
314	1.60	6,7,2,0,0	14
315	1.60	6,7,2,0,0	14

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
316	1.60	6,8,1,0,0	15
317	1.53	7,5,3,0,0	14
318	1.47	8,3,4,0,0	14
319	1.60	6,8,1,0,0	15
320	1.53	8,4,2,1,0	14
321	1.60	6,8,1,0,0	15
322	1.53	7,6,2,0,0	15
323	1.60	7,6,1,1,0	14
324	1.53	7,7,1,0,0	16
325	1.53	8,4,2,1,0	14
326	1.53	8,4,2,1,0	14
327	1.53	8,4,2,1,0	14
328	1.47	8,4,3,0,0	15
329	1.53	7,7,1,0,0	16
330	1.53	7,7,1,0,0	16
331	1.60	7,7,0,1,0	15
332	1.60	7,6,1,1,0	14
333	1.60	6,8,1,0,0	15
334	1.53	7,5,3,0,0	14
335	1.53	7,5,3,0,0	14
336	1.60	6,8,1,0,0	15
337	1.53	7,6,2,0,0	15
338	1.53	8,4,2,1,0	14
339	1.60	7,6,1,1,0	14
340	1.60	6,8,1,0,0	15
341	1.53	7,6,2,0,0	15
342	1.53	8,4,2,1,0	14
343	1.60	6,7,2,0,0	14
344	1.53	7,5,3,0,0	14
345	1.53	7,5,3,0,0	14
346	1.60	6,8,1,0,0	15
347	1.53	8,5,1,1,0	15
348	1.53	8,4,2,1,0	14
349	1.47	8,4,3,0,0	15
350	1.53	7,7,1,0,0	16

ID	δ	$f^{+}_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
351	1.53	7,7,1,0,0	16
352	1.60	7,7,0,1,0	15
353	1.47	8,5,2,0,0	16
354	1.47	8,4,3,0,0	15
355	1.47	8,4,3,0,0	15
356	1.47	8,4,3,0,0	15
357	1.47	9,3,2,1,0	15
358	1.40	9,2,4,0,0	15
359	1.47	8,4,3,0,0	15
360	1.60	7,6,1,1,0	14
361	1.47	8,5,2,0,0	16
362	1.47	8,4,3,0,0	15
363	1.47	8,4,3,0,0	15
364	1.47	9,3,2,1,0	15
365	1.47	8,3,4,0,0	14
366	1.47	8,4,3,0,0	15
367	1.47	8,4,3,0,0	15
368	1.53	8,5,1,1,0	15
369	1.60	6,8,1,0,0	15
370	1.60	6,7,2,0,0	14
371	1.60	7,6,1,1,0	14
372	1.60	6,7,2,0,0	14
373	1.53	7,5,3,0,0	14
374	1.53	7,5,3,0,0	14
375	1.53	8,4,2,1,0	14
376			
377	1.53	7,6,2,0,0	15
378	1.60	6,8,1,0,0	15
379	1.60	7,6,1,1,0	14
380	1.53	7,6,2,0,0	15
381	1.67	5,8,2,0,0	13
382	1.60	6,6,3,0,0	13
383	1.60	6,7,2,0,0	14
384	1.60	6,8,1,0,0	15
385	1.67	5,10,0,0,0	15

ID	δ	$f^{+}_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
386	1.60	6,6,3,0,0	13
387	1.67	5,8,2,0,0	13
388	1.60	6,7,2,0,0	14
389	1.60	6,8,1,0,0	15
390	1.53	7,5,3,0,0	14
391	1.60	6,7,2,0,0	14
392	1.67	5,10,0,0,0	15
393	1.60	6,7,2,0,0	14
394	1.60	7,6,1,1,0	14
395	1.60	7,6,1,1,0	14
396	1.60	6,7,2,0,0	14
397			
398	1.40	9,2,4,0,0	15
399	1.47	8,4,3,0,0	15
400	1.47	9,3,2,1,0	15
401	1.47	8,4,3,0,0	15
402	1.47	8,4,3,0,0	15
403	1.53	8,5,1,1,0	15
404	1.53	7,6,2,0,0	15
405	1.53	7,6,2,0,0	15
406	1.53	7,6,2,0,0	15
407	1.53	7,5,3,0,0	14
408	1.60	6,7,2,0,0	14
409	1.47	8,4,3,0,0	15
410	1.53	7,5,3,0,0	14
411	1.53	7,4,4,0,0	13
412	1.60	6,7,2,0,0	14
413	1.53	7,6,2,0,0	15
414	1.60	6,8,1,0,0	15
415	1.53	7,5,3,0,0	14
416	1.53	7,5,3,0,0	14
417	1.53	8,4,2,1,0	14
418	1.60	6,8,1,0,0	15
419	1.60	7,6,1,1,0	14
420	1.53	7,6,2,0,0	15

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
421	1.53	7,6,2,0,0	15
422	1.53	7,5,3,0,0	14
423	1.60	6,7,2,0,0	14
424	1.47	8,4,3,0,0	15
425	1.47	8,3,4,0,0	14
426	1.60	6,7,2,0,0	14
427	1.53	7,5,3,0,0	14
428	1.60	6,7,2,0,0	14
429	1.60	6,8,1,0,0	15
430	1.60	6,7,2,0,0	14
431	1.60	7,5,2,1,0	13
432	1.60	6,7,2,0,0	14
433	1.60	6,7,2,0,0	14
434	1.60	6,7,2,0,0	14
435	1.60	7,7,0,1,0	15
436	1.60	6,8,1,0,0	15
437	1.53	7,6,2,0,0	15
438	1.47	8,4,3,0,0	15
439	1.47	8,3,4,0,0	14
440	1.53	7,6,2,0,0	15
441	1.47	9,3,2,1,0	15
442	1.53	7,6,2,0,0	15
443	1.53	7,6,2,0,0	15
444	1.60	6,8,1,0,0	15
445	1.60	7,5,2,1,0	13
446	1.60	7,5,2,1,0	13
447	1.53	7,5,3,0,0	14
448	1.53	7,6,2,0,0	15
449	1.60	6,8,1,0,0	15
450	1.60	6,6,3,0,0	13
451	1.60	6,8,1,0,0	15
452	1.60	6,6,3,0,0	13
453	1.67	5,9,1,0,0	14
454	1.60	6,6,3,0,0	13
455	1.60	6,7,2,0,0	14

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
456	1.67	5,9,1,0,0	14
457	1.67	5,9,1,0,0	14
458	1.53	7,6,2,0,0	15
459	1.60	6,6,3,0,0	13
460	1.53	7,4,4,0,0	13
461	1.60	6,7,2,0,0	14
462	1.53	7,5,3,0,0	14
463	1.67	5,9,1,0,0	14
464	1.60	6,7,2,0,0	14
465	1.67	5,9,1,0,0	14
466	1.67	5,9,1,0,0	14
467	1.67	5,7,3,0,0	12
468	1.67	5,9,1,0,0	14
469	1.67	5,8,2,0,0	13
470	1.67	5,9,1,0,0	14
471	1.67	5,8,2,0,0	13
472	1.67	5,8,2,0,0	13
473	1.67	5,10,0,0,0	15
474	1.60	6,7,2,0,0	14
475	1.53	7,6,2,0,0	15
476	1.53	7,5,3,0,0	14
477	1.60	6,7,2,0,0	14
478	1.67	5,10,0,0,0	15
479	1.67	6,7,1,1,0	13
480	1.60	7,6,1,1,0	14
481	1.60	6,8,1,0,0	15
482	1.53	7,6,2,0,0	15
483	1.60	6,7,2,0,0	14
484	1.53	7,5,3,0,0	14
485	1.60	6,7,2,0,0	14
486	1.67	5,10,0,0,0	15
487	1.60	7,6,1,1,0	14
488	1.67	5,9,1,0,0	14
489	1.67	5,9,1,0,0	14
490	1.60	6,8,1,0,0	15

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
491	1.67	5,10,0,0,0	15
492	1.67	6,8,0,1,0	14
493	1.67	6,7,1,1,0	13
494	1.67	5,10,0,0,0	15
495	1.60	6,7,2,0,0	14
496	1.60	6,5,4,0,0	12
497	1.67	5,9,1,0,0	14
498	1.60	6,6,3,0,0	13
499	1.60	6,7,2,0,0	14
500	1.67	5,8,2,0,0	13
501	1.60	6,8,1,0,0	15
502	1.67	5,10,0,0,0	15
503	1.67	5,9,1,0,0	14
504	1.60	6,6,3,0,0	13
505	1.60	6,7,2,0,0	14
506	1.60	6,8,1,0,0	15
507	1.67	5,9,1,0,0	14
508	1.67	6,7,1,1,0	13
509	1.67	6,8,0,1,0	14
510	1.67	5,10,0,0,0	15
511	1.47	9,0,5,1,0	12
512	1.47	8,4,3,0,0	15
513	1.53	7,6,2,0,0	15
514	1.53	7,7,1,0,0	16
515	1.60	7,6,1,1,0	14
516			
517			
518			
519	1.47	8,4,3,0,0	15
520	1.60	7,6,1,1,0	14
521	1.53	7,6,2,0,0	15
522			
523	1.47	8,4,3,0,0	15
524	1.53	8,3,3,1,0	13
525	1.47	8,4,3,0,0	15

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
526	1.47	8,4,3,0,0	15
527	1.53	7,6,2,0,0	15
528	1.60	7,6,1,1,0	14
529	1.60	6,8,1,0,0	15
530	1.60	6,7,2,0,0	14
531	1.60	6,7,2,0,0	14
532	1.67	6,7,1,1,0	13
533	1.53	7,5,3,0,0	14
534	1.60	6,8,1,0,0	15
535	1.60	6,8,1,0,0	15
536	1.60	7,6,1,1,0	14
537			
538	1.47	8,4,3,0,0	15
539	1.40	9,1,5,0,0	14
540	1.53	8,3,3,1,0	13
541	1.53	7,7,1,0,0	16
542	1.53	7,6,2,0,0	15
543	1.53	7,7,1,0,0	16
544	1.60	7,6,1,1,0	14
545	1.60	6,6,3,0,0	13
546	1.60	6,6,3,0,0	13
547	1.60	6,8,1,0,0	15
548	1.53	7,4,4,0,0	13
549	1.60	6,6,3,0,0	13
550	1.60	6,7,2,0,0	14
551	1.53	7,6,2,0,0	15
552	1.67	5,10,0,0,0	15
553	1.60	6,7,2,0,0	14
554	1.53	7,5,3,0,0	14
555	1.60	6,8,1,0,0	15
556	1.53	7,6,2,0,0	15
557	1.60	7,5,2,1,0	13
558	1.67	6,8,0,1,0	14
559	1.60	6,7,2,0,0	14
560	1.67	5,10,0,0,0	15

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
561	1.60	6,7,2,0,0	14
562	1.67	5,9,1,0,0	14
563	1.60	6,8,1,0,0	15
564	1.67	6,6,2,1,0	12
565	1.60	6,7,2,0,0	14
566	1.67	5,10,0,0,0	15
567	1.67	6,7,1,1,0	13
568	1.67	5,10,0,0,0	15
569	1.60	6,7,2,0,0	14
570	1.67	5,8,2,0,0	13
571	1.60	6,7,2,0,0	14
572	1.60	6,6,3,0,0	13
573	1.60	6,7,2,0,0	14
574	1.60	6,7,2,0,0	14
575	1.67	5,9,1,0,0	14
576	1.67	5,9,1,0,0	14
577	1.60	6,6,3,0,0	13
578	1.53	7,5,3,0,0	14
579	1.53	7,5,3,0,0	14
580	1.67	5,9,1,0,0	14
581	1.60	6,8,1,0,0	15
582	1.67	5,10,0,0,0	15
583	1.73	5,8,1,1,0	12
584	1.67	5,9,1,0,0	14
585	1.60	6,7,2,0,0	14
586	1.60	6,7,2,0,0	14
587	1.53	7,5,3,0,0	14
588	1.60	6,8,1,0,0	15
589	1.53	7,5,3,0,0	14
590	1.53	7,4,4,0,0	13
591	1.60	6,7,2,0,0	14
592	1.67	5,9,1,0,0	14
593	1.53	7,5,3,0,0	14
594	1.53	7,5,3,0,0	14
595	1.47	8,3,4,0,0	14

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
596	1.53	7,4,4,0,0	13
597	1.53	7,6,2,0,0	15
598	1.60	6,7,2,0,0	14
599	1.53	7,6,2,0,0	15
600	1.60	6,8,1,0,0	15
601	1.67	5,10,0,0,0	15
602	1.60	6,7,2,0,0	14
603	1.67	6,6,2,1,0	12
604	1.67	5,10,0,0,0	15
605	1.60	6,8,1,0,0	15
606	1.60	6,7,2,0,0	14
607	1.67	6,7,1,1,0	13
608	1.67	5,9,1,0,0	14
609	1.67	5,9,1,0,0	14
610	1.67	5,8,2,0,0	13
611	1.67	5,8,2,0,0	13
612	1.67	5,8,2,0,0	13
613	1.67	5,9,1,0,0	14
614	1.67	5,9,1,0,0	14
615	1.67	5,8,2,0,0	13
616	1.67	5,9,1,0,0	14
617	1.47	8,3,4,0,0	14
618	1.47	8,2,5,0,0	13
619	1.60	6,7,2,0,0	14
620	1.53	7,5,3,0,0	14
621	1.53	7,6,2,0,0	15
622	1.60	6,7,2,0,0	14
623	1.53	7,6,2,0,0	15
624	1.60	6,8,1,0,0	15
625	1.60	6,7,2,0,0	14
626	1.60	6,7,2,0,0	14
627	1.60	6,8,1,0,0	15
628	1.67	5,10,0,0,0	15
629	1.73	5,9,0,1,0	13
630	1.60	6,7,2,0,0	14

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
631	1.67	6,7,1,1,0	13
632	1.67	5,9,1,0,0	14
633	1.60	6,6,3,0,0	13
634	1.60	6,7,2,0,0	14
635	1.67	5,10,0,0,0	15
636	1.67	5,8,2,0,0	13
637	1.67	5,8,2,0,0	13
638	1.60	6,7,2,0,0	14
639	1.67	5,10,0,0,0	15
640	1.60	6,8,1,0,0	15
641	1.60	6,7,2,0,0	14
642	1.60	6,7,2,0,0	14
643	1.60	6,8,1,0,0	15
644	1.60	6,7,2,0,0	14
645	1.73	5,9,0,1,0	13
646	1.67	6,7,1,1,0	13
647	1.67	5,10,0,0,0	15
648	1.67	5,9,1,0,0	14
649	1.67	5,9,1,0,0	14
650	1.73	5,8,1,1,0	12
651	1.67	5,10,0,0,0	15
652	1.67	5,9,1,0,0	14
653	1.60	7,5,2,1,0	13
654	1.60	6,8,1,0,0	15
655	1.60	6,8,1,0,0	15
656	1.67	6,8,0,1,0	14
657	1.60	7,7,0,1,0	15
658	1.40	9,3,3,0,0	16
659	1.47	9,3,2,1,0	15
660	1.47	9,3,2,1,0	15
661	1.40	9,3,3,0,0	16
662	1.60	7,7,0,1,0	15
663	1.53	7,7,1,0,0	16
664	1.53	7,7,1,0,0	16
665	1.53	8,4,2,1,0	14

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
666	1.47	8,4,3,0,0	15
667	1.53	8,4,2,1,0	14
668	1.47	8,4,3,0,0	15
669	1.60	7,7,0,1,0	15
670	1.60	7,7,0,1,0	15
671	1.53	7,7,1,0,0	16
672	1.53	8,4,2,1,0	14
673	1.53	8,4,2,1,0	14
674			
675	1.47	8,4,3,0,0	15
676	1.53	7,7,1,0,0	16
677	1.53	7,7,1,0,0	16
678	1.60	7,7,0,1,0	15
679	1.47	8,4,3,0,0	15
680	1.53	8,4,2,1,0	14
681	1.53	8,4,2,1,0	14
682	1.47	8,4,3,0,0	15
683	1.53	7,7,1,0,0	16
684	1.60	7,7,0,1,0	15
685	1.53	7,7,1,0,0	16
686	1.60	7,7,0,1,0	15
687	1.67	5,10,0,0,0	15
688	1.67	6,7,1,1,0	13
689	1.60	6,8,1,0,0	15
690	1.67	6,8,0,1,0	14
691	1.67	6,7,1,1,0	13
692	1.67	5,10,0,0,0	15
693	1.67	5,10,0,0,0	15
694	1.67	6,8,0,1,0	14
695	1.60	6,5,4,0,0	12
696	1.60	6,7,2,0,0	14
697	1.67	5,8,2,0,0	13
698	1.60	6,8,1,0,0	15
699	1.67	5,9,1,0,0	14
700	1.53	7,5,3,0,0	14

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
701	1.60	6,7,2,0,0	14
702	1.60	6,7,2,0,0	14
703	1.53	7,5,3,0,0	14
704	1.67	5,9,1,0,0	14
705	1.53	7,6,2,0,0	15
706	1.60	6,6,3,0,0	13
707	1.60	6,7,2,0,0	14
708	1.53	7,6,2,0,0	15
709	1.60	6,6,3,0,0	13
710	1.53	7,5,3,0,0	14
711	1.60	6,7,2,0,0	14
712	1.67	5,9,1,0,0	14
713	1.60	6,8,1,0,0	15
714	1.60	6,6,3,0,0	13
715	1.60	6,6,3,0,0	13
716	1.60	6,8,1,0,0	15
717	1.67	5,8,2,0,0	13
718	1.60	6,6,3,0,0	13
719	1.67	5,9,1,0,0	14
720	1.67	5,9,1,0,0	14
721	1.60	6,6,3,0,0	13
722	1.53	7,6,2,0,0	15
723	1.53	7,6,2,0,0	15
724	1.53	7,4,4,0,0	13
725	1.60	6,6,3,0,0	13
726	1.60	6,7,2,0,0	14
727	1.60	6,8,1,0,0	15
728	1.67	5,9,1,0,0	14
729	1.67	5,10,0,0,0	15
730	1.67	5,8,2,0,0	13
731	1.67	5,7,3,0,0	12
732	1.67	5,8,2,0,0	13
733	1.67	5,7,3,0,0	12
734	1.67	5,9,1,0,0	14
735	1.67	5,10,0,0,0	15

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
736	1.67	5,9,1,0,0	14
737	1.67	5,9,1,0,0	14
738	1.60	6,6,3,0,0	13
739	1.60	6,7,2,0,0	14
740	1.67	5,10,0,0,0	15
741	1.60	6,8,1,0,0	15
742	1.67	6,7,1,1,0	13
743	1.67	5,9,1,0,0	14
744	1.67	6,8,0,1,0	14
745	1.53	7,6,2,0,0	15
746	1.60	6,8,1,0,0	15
747	1.53	7,5,3,0,0	14
748	1.47	8,3,4,0,0	14
749	1.47	8,4,3,0,0	15
750	1.60	7,6,1,1,0	14
751	1.60	6,8,1,0,0	15
752	1.67	6,8,0,1,0	14
753	1.60	6,7,2,0,0	14
754	1.67	5,9,1,0,0	14
755	1.67	5,9,1,0,0	14
756	1.60	6,7,2,0,0	14
757	1.60	6,8,1,0,0	15
758	1.67	6,7,1,1,0	13
759	1.67	5,9,1,0,0	14
760	1.67	6,8,0,1,0	14
761	1.67	5,9,1,0,0	14
762	1.60	6,7,2,0,0	14
763	1.60	6,8,1,0,0	15
764	1.73	5,9,0,1,0	13
765	1.67	6,7,1,1,0	13
766	1.53	7,6,2,0,0	15
767	1.53	7,5,3,0,0	14
768	1.53	7,5,3,0,0	14
769	1.47	8,4,3,0,0	15
770	1.53	7,6,2,0,0	15

ID	δ	$f_{2,i}^{+}$ $(3 \le i \le 7)$	f_1^0
771	1.60	7,6,1,1,0	14
772	1.67	6,8,0,1,0	14
773	1.60	6,8,1,0,0	15
774	1.53	7,6,2,0,0	15
775	1.60	6,7,2,0,0	14
776	1.60	6,8,1,0,0	15
777	1.60	6,7,2,0,0	14
778	1.60	6,7,2,0,0	14
779	1.67	6,7,1,1,0	13
780	1.60	7,6,1,1,0	14
781	1.60	6,8,1,0,0	15
782	1.67	5,9,1,0,0	14
783	1.60	6,6,3,0,0	13
784	1.60	6,7,2,0,0	14
785	1.67	5,10,0,0,0	15
786	1.60	6,8,1,0,0	15
787	1.73	5,9,0,1,0	13
788	1.60	6,8,1,0,0	15
789	1.67	6,7,1,1,0	13
790	1.47	8,3,4,0,0	14
791	1.53	7,5,3,0,0	14
792	1.60	6,8,1,0,0	15
793	1.47	8,4,3,0,0	15
794	1.53	7,6,2,0,0	15
795	1.60	7,6,1,1,0	14
796	1.67	6,8,0,1,0	14
797	1.60	6,8,1,0,0	15
798	1.47	8,3,4,0,0	14
799	1.40	9,2,4,0,0	15
800	1.53	7,6,2,0,0	15
801	1.47	8,4,3,0,0	15
802	1.53	7,6,2,0,0	15
803	1.60	7,6,1,1,0	14
804	1.53	8,5,1,1,0	15
805	1.60	6,8,1,0,0	15

ID	δ	$f_{2,i}^{+}$ $(3 \le i \le 7)$	f_1^0
806	1.53	7,6,2,0,0	15
807	1.53	7,4,4,0,0	13
808	1.60	6,7,2,0,0	14
809	1.60	6,8,1,0,0	15
810	1.67	6,7,1,1,0	13
811	1.60	7,6,1,1,0	14
812	1.67	5,10,0,0,0	15
813	1.60	6,8,1,0,0	15
814	1.67	5,9,1,0,0	14
815	1.67	5,8,2,0,0	13
816	1.67	5,9,1,0,0	14
817	1.67	5,10,0,0,0	15
818	1.67	5,9,1,0,0	14
819	1.73	5,9,0,1,0	13
820	1.73	5,9,0,1,0	13
821	1.67	5,9,1,0,0	14
822	1.67	5,9,1,0,0	14
823	1.60	6,8,1,0,0	15
824	1.67	5,9,1,0,0	14
825	1.60	6,7,2,0,0	14
826	1.60	6,7,2,0,0	14
827	1.67	6,7,1,1,0	13
828	1.67	6,8,0,1,0	14
829	1.67	5,9,1,0,0	14
830	1.60	6,8,1,0,0	15
831	1.60	6,6,3,0,0	13
832	1.67	5,9,1,0,0	14
833	1.60	6,7,2,0,0	14
834	1.67	6,7,1,1,0	13
835	1.67	5,10,0,0,0	15
836	1.67	6,8,0,1,0	14
837	1.67	5,9,1,0,0	14
838	1.73	5,9,0,1,0	13
839	1.67	5,10,0,0,0	15
840	1.40	9,3,3,0,0	16

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
841	1.60	7,7,0,0,1	14
842	1.53	7,7,1,0,0	16
843	1.47	8,4,3,0,0	15
844			
845	1.53	7,7,1,0,0	16
846	1.60	7,7,0,0,1	14
847	1.53	7,5,3,0,0	14
848	1.60	6,8,1,0,0	15
849	1.60	6,8,1,0,0	15
850	1.60	6,8,1,0,0	15
851	1.60	7,7,0,0,1	14
852	1.47	8,4,3,0,0	15
853	1.47	8,4,3,0,0	15
854	1.53	7,7,1,0,0	16
855	1.53	7,7,1,0,0	16
856	1.60	7,7,0,0,1	14
857	1.40	9,1,5,0,0	14
858	1.47	8,4,3,0,0	15
859	1.53	7,7,1,0,0	16
860	1.53	7,7,1,0,0	16
861	1.60	7,7,0,0,1	14
862	1.60	6,8,1,0,0	15
863	1.47	8,4,3,0,0	15
864	1.53	7,5,3,0,0	14
865	1.60	7,6,1,1,0	14
866	1.67	5,10,0,0,0	15
867	1.60	6,8,1,0,0	15
868	1.67	5,9,1,0,0	14
869	1.60	6,6,3,0,0	13
870	1.60	6,8,1,0,0	15
871	1.67	5,9,1,0,0	14
872	1.67	6,7,1,1,0	13
873	1.67	6,7,1,1,0	13
874	1.67	5,10,0,0,0	15
875	1.60	6,6,3,0,0	13

ID	δ	$f^+_{2,i}$ $(3 \leq i \leq 7)$	f^0_1
876	1.60	6,7,2,0,0	14
877	1.60	6,8,1,0,0	15
878	1.73	5,9,0,1,0	13
879	1.67	5,10,0,0,0	15
880	1.60	6,7,2,0,0	14
881	1.53	7,6,2,0,0	15
882	1.60	6,7,2,0,0	14
883	1.53	7,5,3,0,0	14
884	1.53	7,6,2,0,0	15
885	1.67	6,7,1,1,0	13
886	1.67	6,8,0,1,0	14
887	1.67	5,10,0,0,0	15
888	1.53	7,4,4,0,0	13
889	1.60	6,7,2,0,0	14
890	1.53	7,6,2,0,0	15
891	1.60	6,8,1,0,0	15
892	1.53	7,6,2,0,0	15
893	1.67	6,7,1,1,0	13
894	1.67	6,8,0,1,0	14
895	1.67	5,10,0,0,0	15
896	1.67	5,8,2,0,0	13
897	1.67	5,8,2,0,0	13
898	1.67	5,9,1,0,0	14
899	1.67	5,10,0,0,0	15
900	1.73	5,8,1,1,0	12
901	1.67	5,10,0,0,0	15
902	1.73	5,9,0,1,0	13
903	1.67	5,10,0,0,0	15
904	1.67	5,10,0,0,0	15
905	1.60	6,7,2,0,0	14
906	1.60	6,8,1,0,0	15
907	1.67	6,8,0,0,1	13
908	1.67	5,10,0,0,0	15
909	1.60	6,7,2,0,0	14
910	1.60	6,7,2,0,0	14

ID	δ	$f_{2,i}^{+}$ $(3 \leq i \leq 7)$	f_1^0
911	1.67	5,10,0,0,0	15
912	1.60	6,8,1,0,0	15
913	1.60	6,8,1,0,0	15
914	1.67	5,10,0,0,0	15
915	1.67	6,8,0,0,1	13
916	1.67	5,10,0,0,0	15
917	1.67	5,9,1,0,0	14
918	1.67	5,9,1,0,0	14
919	1.67	5,10,0,0,0	15
920	1.73	5,9,0,0,1	12
921	1.67	5,10,0,0,0	15
922	1.67	5,10,0,0,0	15

C.3 Plane arrangements with 6 planes

We use $F_{3,i}^{+}(\mathcal{A}_{3,6})$ and $f_{3,i}^{+}(\mathcal{A}_{3,6})$ $(4 \leq i \leq 6)$ to denote the type and number of bounded cells (simple 3-polytopes) with i facets respectively. Obviously, $F_{3,4}^{+}(\mathcal{A}_{3,6})$ is tetrahedron and $F_{3,5}^{+}(\mathcal{A}_{3,6})$ is simplex prism. There are 2 combinatorial types of $F_{3,6}^{+}(\mathcal{A}_{3,6})$: cube and 6-shell, which are denoted by $F_{3,6C}^{+}(\mathcal{A}_{3,6})$ and $F_{3,6S}^{+}(\mathcal{A}_{3,6})$ respectively. Note that $\sum_{i=4}^{6} f_{3,i}^{+}(\mathcal{A}_{3,6}) = f_3^{+}(\mathcal{A}_{3,6}) = 10$.

ID	δ	$f^+_{3,4}, f^+_{3,5}, f^+_{3,6C}, f^+_{3,6S}$	f^0_2
1	1.8	5,2,0,3	24
2	1.8	4,4,0,2	24
3	1.8	3,6,1,0	24
4	1.8	3,6,1,0	24
5	1.9	3,5,0,2	23
6	1.9	3,5,1,1	23
7	1.8	3,6,1,0	24
8	1.8	4,4,0,2	24
9	1.8	4,4,0,2	24
10	1.8	3,6,1,0	24
11	1.8	3,6,1,0	24
12	1.9	4,3,0,3	23
13	1.9	4,3,0,3	23
14	1.9	3,5,0,2	23
15	1.9	3,5,0,2	23
16	1.8	5,2,0,3	24
17	1.7	5,3,0,2	25
18	1.7	5,3,0,2	25
19	1.8	4,4,0,2	24
20	1.8	4,4,1,1	24
21	1.9	4,3,0,3	23
22	1.9	3,5,1,1	23

ID	δ	$f^+_{3,4}, f^+_{3,5}, f^+_{3,6C}, f^+_{3,6S}$	f^0_2
23	1.8	3,6,1,0	24
24	1.8	5,2,0,3	24
25	1.8	4,4,1,1	24
26	1.9	4,3,0,3	23
27	1.8	3,6,1,0	24
28	1.8	3,6,1,0	24
29	**2.0**	**3,4,0,3**	**22**
30	1.9	3,5,1,1	23
31	1.8	3,6,1,0	24
32	1.8	4,4,0,2	24
33	1.8	4,4,0,2	24
34	1.9	3,5,1,1	23
35	1.9	3,5,0,2	23
36	1.8	3,6,1,0	24
37	1.8	4,4,1,1	24
38	1.8	3,6,1,0	24
39	**2.0**	**3,4,0,3**	**22**
40	1.8	3,6,1,0	24
41	1.9	3,5,1,1	23
42	1.8	3,6,1,0	24
43	1.8	3,6,1,0	24

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