An Example of Two Phase Simplex Method

AdvOL @McMaster, http://optlab.mcmaster.ca

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Consider the following LP problem.

$$\begin{array}{lrl} \max & z = 2x_1 + 3x_2 + x_3 & \\ \text{s.t.} & x_1 + x_2 + x_3 & \leq 40 \\ & 2x_1 + x_2 - x_3 & \geq 10 \\ & -x_2 + x_3 & \geq 10 \\ & x_1, x_2, x_3 & \geq 0 \end{array}$$

It can be transformed into the standard form by introducing 3 slack variables x_4, x_5 and x_6.

$$\begin{array}{lll} \max & z = 2x_1 + 3x_2 + x_3 & \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 & = 40 \\ & 2x_1 + x_2 - x_3 \quad - x_5 & = 10 \\ & -x_2 + x_3 \quad - x_6 & = 10 \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0 \end{array}$$

There is no obvious initial basic feasible solution, and it is not even known whether there exists one. We can use Phase I method to find out. Consider the following LP problem derived from the original one by relaxing the second and third constraints and introducing a new objective function.

$$\begin{array}{lll} \min & x_7 + x_8, \quad (\text{or } \max w = -x_7 - x_8) & \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 & = 40 \\ & 2x_1 + x_2 - x_3 \quad - x_5 \quad + x_7 & = 10 \\ & -x_2 + x_3 \quad - x_6 \quad + x_8 & = 10 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 & \geq 0 \end{array}$$

This problem (Phase I) has an initial basic feasible solution with basic variables being x_4, x_7 and x_8. If the minimum value of $x_7 + x_8$ is 0, then both x_7 and x_8 are 0. As the result, the optimal solution of the Phase I problem is an basic feasible solution of the original problem. If the minimum value of $x_7 + x_8$ is bigger than 0, then the original problem is not feasible. We construct tableaus to solve the Phase I problem. The objective value w should be written in terms of non-basic variables:

$$w = -x_7 - x_8 = -20 + 2x_1 - x_5 - x_6.$$

The initial tableau is shown below (the basic variables are shown in bold font).

w	x_1	x_2	x_3	$\boldsymbol{x_4}$	x_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$		
1	-2	0	0	0	1	1	0	0	=	-20
0	1	1	1	1	0	0	0	0	=	40
0	2	1	-1	0	-1	0	1	0	=	10
0	0	-1	1	0	0	-1	0	1	=	10

The entering and leaving variables would be x_1 and x_7 respectively:

w	$\boldsymbol{x_1}$	x_2	x_3	$\boldsymbol{x_4}$	x_5	x_6	x_7	$\boldsymbol{x_8}$		
1	0	1	-1	0	0	1	1	0	=	-10
0	0	0.5	1.5	1	0.5	0	-0.5	0	=	35
0	1	0.5	-0.5	0	-0.5	0	0.5	0	=	5
0	0	-1	1	0	0	-1	0	1	=	10

The entering and leaving variables would be x_3 and x_8 respectively:

w	$\boldsymbol{x_1}$	x_2	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	x_5	x_6	x_7	x_8		
1	0	1	0	0	0	0	1	1	=	0
0	0	2	0	1	0.5	1.5	-0.5	-1.5	=	20
0	1	0	0	0	-0.5	-0.5	0.5	0.5	=	10
0	0	-1	1	0	0	-1	0	1	=	10

The optimal value of the Phase I problem is $w = 0$. So the original problem is feasible, and a basic feasible solution is $x_1 = 10, x_3 = 10, x_4 = 20, x_2 = x_5 = x_6 = 0$. Now we can start Phase II. Again the objective value z should be represented by the non-basic variables:

$$z = 2x_1 + 3x_2 + x_3 = 30 + 4x_2 + x_5 + 2x_6.$$

The initial tableau is (the last Phase I tableau with x_7 and x_8 taken away):

z	$\boldsymbol{x_1}$	x_2	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	x_5	x_6		
1	0	-4	0	0	-1	-2	=	30
0	0	2	0	1	0.5	1.5	=	20
0	1	0	0	0	-0.5	-0.5	=	10
0	0	-1	1	0	0	-1	=	10

The entering and leaving variables would be x_2 and x_4 respectively:

z	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_4	x_5	x_6		
1	0	0	0	2	0	1	=	70
0	0	1	0	0.5	0.25	0.75	=	10
0	1	0	0	0	-0.5	-0.5	=	10
0	0	0	1	0.5	0.25	-0.25	=	20

Thus, the optimal value $z = 70$, and the optimal solution is $x_1 = x_2 = 10, x_3 = 20, x_4 = x_5 = x_6 = 0$.