# An Example of Degeneracy in Linear Programming

An LP is degenerate if in a basic feasible solution, one of the basic variables takes on a zero value. Degeneracy is caused by redundant constraint(s) and could cost simplex method extra iterations, as demonstrated in the following example.

$$\begin{array}{lrl} \max & z = x_1 + x_2 + x_3 & \\ \text{s.t.} & x_1 + x_2 & \leq 1 \\ & -x_2 + x_3 & \leq 0 \\ & x_1, x_2, x_3 & \geq 0 \end{array}$$

Note that constraint $x_2 \geq 0$ follows from constraints $-x_2 + x_3 \leq 0$ and $x_3 \geq 0$, and is thus redundant. The feasible region, a 3D polytope, is shown in figure 1. Notice that the origin is a vertex of the polytope. It can be transformed into the standard form by introducing two slack variables $x_4$ and $x_5$.

$$\begin{array}{lrl} \max & z = x_1 + x_2 + x_3 & \\ \text{s.t.} & x_1 + x_2 + x_4 & = 1 \\ & -x_2 + x_3 + x_5 & = 0 \\ & x_1, x_2, x_3, x_4, x_5 & \geq 0 \end{array}$$

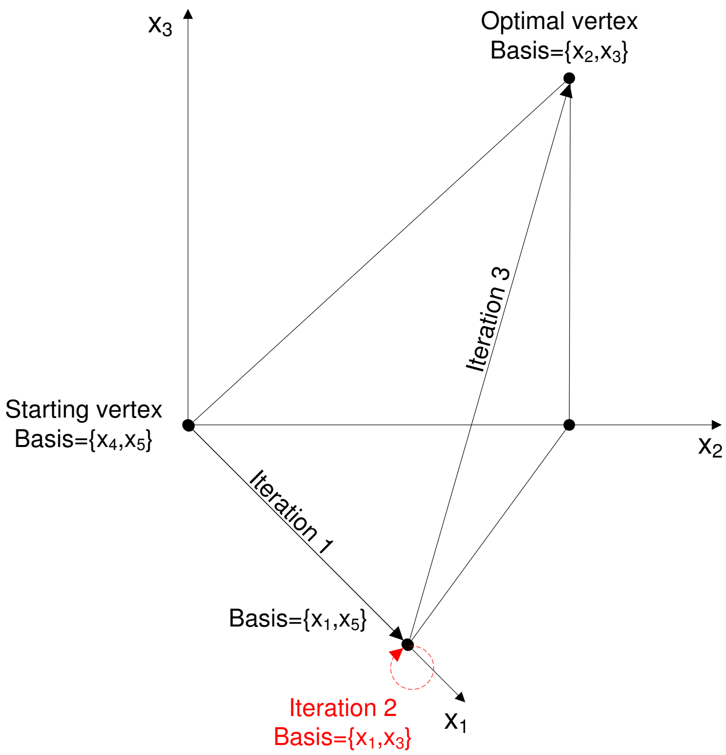


Figure 1: Geometric view of simplex method

We can use the origin as the starting point of simplex method, which corresponds to basic feasible solution: $\{x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1, x_5 = 0\}$. The tableau is shown below (the basic variables are shown in bold font).

| Variables | $z$ | $x_1$ | $x_2$ | $x_3$ | $\boldsymbol{x_4}$ | $\boldsymbol{x_5}$ | | |
|---|---|---|---|---|---|---|---|---|
| Coefficients | 1 | -1 | -1 | -1 | **0** | **0** | = | 0 |
| | 0 | 1 | 1 | 0 | **1** | **0** | = | 1 |
| | 0 | 0 | -1 | 1 | **0** | **1** | = | 0 |
| Values | 0 | 0 | 0 | 0 | **1** | **0** | | |

Note that value of basic variable $x_5$ is 0. For the first iteration of simplex method, there are 3 choices of entering variables: $x_1, x_2, x_3$, corresponding to the 3 edges of the polytope that connect the origin to other vertices. We choose $x_1$. (In practice, the choice of entering variable is determined by the pivot rule used.) The leaving variable should be $x_4$.

| Variables | $z$ | $\boldsymbol{x_1}$ | $x_2$ | $x_3$ | $x_4$ | $\boldsymbol{x_5}$ | | |
|---|---|---|---|---|---|---|---|---|
| Coefficients | 1 | **0** | 0 | -1 | 1 | **0** | = | 1 |
| | 0 | **1** | 1 | 0 | 1 | **0** | = | 1 |
| | 0 | **0** | -1 | 1 | 0 | **1** | = | 0 |
| Values | 1 | **1** | 0 | 0 | 0 | **0** | | |

The value of the objective function ($z$) is increased to 1 after the first iteration. In iteration 2, the only choice of entering variable is $x_2$. The leaving variable should be $x_1$.

| Variables | $z$ | $\boldsymbol{x_1}$ | $x_2$ | $\boldsymbol{x_3}$ | $x_4$ | $x_5$ | | |
|---|---|---|---|---|---|---|---|---|
| Coefficients | 1 | **0** | -1 | **0** | 1 | 1 | = | 1 |
| | 0 | **1** | 1 | **0** | 1 | 0 | = | 1 |
| | 0 | **0** | -1 | **1** | 0 | 1 | = | 0 |
| Values | 1 | **1** | 0 | **0** | 0 | 0 | | |

After iteration 2, the value of the objective function remains the same ($z = 1$). Due to degeneracy, basis change does not cause the iteration to follow an edge; we are still in the same vertex (see figure 1). After the last iteration shown below, the optimal value ($z = 2$) is found.

| Variables | $z$ | $x_1$ | $\boldsymbol{x_2}$ | $\boldsymbol{x_3}$ | $x_4$ | $x_5$ | | |
|---|---|---|---|---|---|---|---|---|
| Coefficients | 1 | 1 | **0** | **0** | 2 | 1 | = | 2 |
| | 0 | 1 | **1** | **0** | 1 | 0 | = | 1 |
| | 0 | 1 | **0** | **1** | 1 | 1 | = | 1 |
| Values | 2 | 0 | **1** | **1** | 0 | 0 | | |